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BRANCHING BROWNIAN MOTION WITH ABSORPTION

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We consider a branching diffusion $\{Z_t\}_{t\geq 0}$ in which particles move during their life time according to a Brownian motion with drift $-\mu$ and variance coefficient σ^2, and in which each particle which enters the negative half line is instantaneously removed from the population. If particles die with probability $c\,\mathrm{d}t+\mathrm{o}(\mathrm{d}t)$ in $[t, t+\mathrm{d}t]$ and if the mean number of offspring per particle is $m>1$, then Z_t dies out w.p.l. if $\mu\geq\mu_0=\{2\sigma^2c(m-1)\}^{1/2}$. If $\mu<\mu_0$, then Z_t grows exponentially with positive probability. Our main concern here is with the critical case where $\mu=\mu_0$. Even though $\mathbf{E}\{Z_T\}\sim\text{const. }T^{-3/2}$ in this case, we find that $\mathbf{P}\{Z_T>0\}$ is only $\exp\{-\text{const. }T^{1/3}+\mathrm{o}(\log T)^2\}$, and conditionally on $\{Z_T>0\}$ there are with high probability much fewer particles alive at time T than $\mathbf{E}\{Z_T\mid Z_T>0\}$.

branching Brownian motion	limit theorem for branching process
branching process	survival probability
absorption	growth rate

1. Introduction and statement of results

We consider a branching diffusion on the real line of the following form: Each particle alive at time t dies in $[t, t+\mathrm{d}t]$ with probability $c\,\mathrm{d}t+\mathrm{o}(\mathrm{d}t)$. If it dies it gives birth to a random number of particles; the distribution of this number of offspring is denoted by G. c and G are the same for all particles and independent of the position of the particle. The offspring of any particle J is born at the position where J dies. Between its birth and its death a particle moves according to a Brownian motion with drift $-\mu$ (*note the minus sign here*) and (constant) variance coefficient σ^2. Any particle which enters $(-\infty, 0]$ is removed instantaneously from the population and does not produce any offspring. All particles move, die and produce offspring independently of each other.

We shall use the following notation: $Z_t(\cdot)$ is the integer valued measure counting the number of particles alive at time t in Borel sets, i.e.,

$$Z_t(A)=\text{number of particles alive at time } t \text{ with position in } A$$

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(A a Borel set of $\mathbf{R}$; note that $Z_t((-\infty, 0])=0$ for all t because particles entering $(-\infty, 0]$ are removed),

$$Z_t = Z_t(\mathbf{R}) = \text{number of particles alive at time } t,$$

$$m = \text{mean number of offspring per particle} = \int_0^\infty x\,\mathrm{d}G(x),$$

$$b = \int x(x-1)\,\mathrm{d}G(x),$$

$\mathbf{P}^x$ will denote the probability measure for the $\{Z_t(\cdot)\}$ process, given that we start at $t=0$ with one particle at x, i.e., $Z_0(A) = I_A(x)$, where I_A is the indicator function of A. $\mathbf{E}^x$ denotes expectation w.r.t. $\mathbf{P}^x$.

Throughout K_i will denote various constants whose precise value is of little importance. At different appearances K_i may take different values. However, K_i always satisfies $0<K_i<\infty$. Usually K_i depends only on μ, σ, b, c and m, but in Section 2 we shall allow it to depend on some additional parameters; when such dependence on extra parameters occurs it will be stated explicitly.

A careful mathematical description and construction of the probability space for such branching diffusions can be found in Ikeda et al. [6]. Here we shall take it for granted that we have a process $\{Z_t(\cdot)\}_{t\geq 0}$ with the above properties and that it has the strong Markov property with respect to the σ-fields

$$\mathcal{H}_t = \sigma\text{-field generated by } \{Z_s(\cdot): 0 \leq s \leq t\}.$$

The first theorem delineates the regions where the process is super critical, critical or subcritical and gives the rate of growth in the supercritical case. The distinction between the cases is easy and can readily be obtained from [4] and [10]. Also the asymptotic behavior (1.1)–(1.3) of $\mathbf{E}\, Z_T$ is easy to obtain from Lemma 3.1 and standard estimates for Brownian motion. So far we have only an ugly and complicated proof for the growth results (1.5)–(1.8) in the supercritical case, and we shall therefore not prove Theorem 1.1 here. Note that (1.5) was already proved by S. Watanabe [13] in the case $\mu = 0$, Δ compact.

Theorem 1.1. *If $m<\infty$ then there exists a $0<C=C(x,\mu,\sigma)<\infty$ such that for $x>0$ and $T\to\infty$*

$$\mathbf{E}^x Z_T \sim C T^{-3/2} \exp\left\{c(m-1) - \frac{\mu^2}{2\sigma^2}\right\} T \quad \text{if } \mu>0, \tag{1.1}$$

$$\mathbf{E}^x Z_T \sim C T^{-1/2} \exp c(m-1)T \qquad \text{if } \mu=0 \tag{1.2}$$

and

$$\mathbf{E}^x Z_T \sim C \exp c(m-1)T \qquad \text{if } \mu<0. \tag{1.3}$$

Consequently, if $m>1$ and $\mu \geq \mu_0 \equiv \{2\sigma^2 c(m-1)\}^{1/2}$, then

$$\mathbf{P}^x\{Z_T>0\} \leq \mathbf{E}^x Z_T \to 0 \tag{1.4}$$

and Z_T dies out eventually w.p.l. If $\mu < \mu_0$, $m > 1$ and $b < \infty$ then Z_T dies out or grows at the rate prescribed by $\mathbf{E}^x Z_T$. More precisely, for $0 \le \mu < \mu_0$ there exists a random variable w such that a.e. $[\mathbf{P}^x]$ simultaneously for every interval $\Delta \subset (0, \infty)$ (including semi-infinite intervals)

$$\lim_{T\to\infty} \frac{Z_T(\Delta)}{\mathbf{E}^x Z_T(\Delta)} = w, \tag{1.5}$$

$$\mathbf{P}^x\{Z_T \text{ does not die out}\} > 0 \quad \textit{for } x > 0, \tag{1.6}$$

and

$$w > 0 \quad \textit{a.e. } [\mathbf{P}^x] \textit{ on the set } \{Z_T \text{ does not die out}\}. \tag{1.7}$$

If $\mu < 0$, $m > 1$ and $b < \infty$ then there exists a random variable w such that

$$\mathrm{e}^{-Tc(m-1)} Z_T \to w \quad \textit{a.e. } [\mathbf{P}^x] \tag{1.8}$$

and such that (1.6) *and* (1.7) *hold.*

Remark 1.2. The denominator $\mathbf{E}^x Z_T(\Delta)$ in (1.5) behaves like constant. $T^{-3/2} \exp\{c(m-1) - \frac{1}{2}\mu^2/\sigma^2\}T$ for any bounded interval $\Delta \subset (0, \infty)$ if $0 \le \mu < \mu_0$. This same behavior prevails for semi-infinite Δ if $0 < \mu < \mu_0$. However, for $\mu = 0$ and $\Delta = (a, \infty)$, $\mathbf{E}^x Z_T((a, \infty))$ is constant $\cdot\ T^{-1/2} \exp c(m-1)T$.

We next turn to the critical case where we find somewhat unexpectedly that $\mathbf{P}\{Z_T > 0\}$ is of a completely different order of magnitude than $\mathbf{E}\, Z_T$ (which behaves like $T^{-3/2}$ by (1.1)). Also, given $\{Z_T > 0\}$ the distribution of Z_T is not at all concentrated near $\mathbf{E}\{Z_T \mid Z_T > 0\}$. We can view $Z_T(\cdot)$ as a branching process with infinitely many types by interpreting the position of a particle as its type. If the expectation operator has good spectral properties, then it is known (see [5]) that $\mathbf{P}\{Z_T > 0\}$ is of the same order as $\mathbf{E}\{Z_T\}$ and [5] even has Yaglom limit theorems for the conditional distribution of Z_T, given $\{Z_T > 0\}$. The new phenomena arise here because $(0, \infty)$ is not compact and even though most particles enter $(-\infty, 0]$ and are removed, a significant number moves far out to the right. It is still open what the analogue of Yaglom's theorem is, i.e., what the proper limit theorem is for $Z_T(\cdot)$ given that $\{Z_T > 0\}$. (1.11) and (1.12) only give partial information in this direction.

Theorem 1.3. *Let $m > 1$, $b < \infty$ and*

$$\mu = \mu_0 \equiv \{2\sigma^2 c(m-1)\}^{1/2}. \tag{1.9}$$

Then there exist constants $0 < K_1 - K_3 < \infty$ depending on σ, b, c and m only such that for $x > 0$

$$x \exp\left\{\frac{\mu}{\sigma^2} x - K_1 (\log T)^2\right\} \le \mathbf{P}^x\{Z_T > 0\} \exp\left(\frac{3\mu^2\pi^2}{2\sigma^2}\right)^{1/3} T^{1/3}$$

$$\le (1+x) \exp\left\{\frac{\mu}{\sigma^2} x + K_1 (\log T)^2\right\}. \tag{1.10}$$

Moreover, as $T \to \infty$,

$$\mathbf{P}^x\{Z_T([K_2 T^{2/9}(\log T)^{2/3}, \infty)) > 0 \mid Z_T > 0\} \to 0 \tag{1.11}$$

and

$$\mathbf{P}^x\{Z_T \geq \exp K_3 T^{2/9}(\log T)^{2/3} \mid Z_T > 0\} \to 0. \tag{1.12}$$

The not very complicated proof of Theorem 1.3 is given in Section 3. Unfortunately it takes a very long Section 2 to develop estimates for Brownian motion absorption probabilities. These estimates (Harnack type inequalities and estimates of probabilities to stay between two curves) seem interesting by themselves, but for the reader interested in branching diffusions only they are mere technical tools and he should skip their proofs.

2. Estimates for Brownian motion restricted to a region

Throughout this section $\{W(t)\}_{t\geq 0}$ is a Brownian motion with drift $-\mu$ and variance coefficient σ^2. The initial position, $W(0)$, is not restricted to be zero, but may be any real number. In other words, we consider W as a Markov process with state space Ω = the continuous functions on $[0, \infty)$, $W(t, \omega) = (\omega' t)$ for $\omega \in \Omega$, and with the family of measures $\mathbf{P}^x$ [2] with the properties

(i) $\mathbf{P}^x\{W(0) = x\} = 1$,

(ii) For every $0 \leq t_1 < t_2 < \cdots < t_n$, $W(t_{i+1}) - W(t_i)$, $1 \leq i \leq n-1$, are independent under $\mathbf{P}^x$ and normalty distributed with mean $-\mu(t_{i+1} - t_i)$ and variance $\sigma^2(t_{i+1} - t_i)$. This Markov process has the strong Markov property and we refer the reader to [8, Chapter 1] for the complete description of this setup. As usual $\mathbf{E}^x$ denotes the expectation w.r.t. $\mathbf{P}^x$.

We shall make frequent use of the following known facts (see [1, Section 4.3]): Under $\mathbf{P}^x$, and given $W(t) = y$, the random function $s \to W(s) - x - t^{-1}s(y - x)$, $0 \leq s \leq t$, is a Brownian bridge (up to changes in the time and space scale) and the random function

$$s \to V(s) \equiv \frac{t+s}{t\sigma}\left[W\left(\frac{st}{t+s}\right) - x - \frac{s}{t+s}(y-x)\right], \quad s \geq 0, \tag{2.1}$$

is a standard Brownian motion starting at $V(0) = 0$. Whenever $a < x$, $a + bt < y$,

$$\begin{aligned} &\mathbf{P}^x\{W(s) \leq a + bs \text{ for some } 0 \leq s \leq t \mid W(t) = y\} \\ &\qquad = \mathbf{P}\left\{V(s) < \frac{a-x}{\sigma} + \frac{a+bt-y}{t\sigma}s \text{ for some } s \geq 0\right\} \\ &\qquad = \exp -\frac{2}{t\sigma^2}(x-a)(y-a-bt). \end{aligned} \tag{2.2}$$

[2] As stated in the introduction we also use $\mathbf{P}^x$ for conditional probabilities for the $Z_t(\cdot)$ process, given $Z_0(A) = I_A(x)$. This latter interpretation will not occur before Section 3, though.

Note that (2.2) does not involve μ, and in fact this relation is valid for any value of μ.

We now fix a $T>0$ and consider two continuous functions f and g on $[0, T]$ satisfying

$$f(s)<g(s) \quad \text{on } [0, T]. \tag{2.3}$$

We introduce the stopping times[3]

$$\begin{aligned} \sigma_{\mathrm{L}} &= \inf\{0 \leqslant s \leqslant T\colon W(s) \leqslant f(s)\}, \\ \sigma_{\mathrm{U}} &= \inf\{0 \leqslant s \leqslant T\colon W(s) \geqslant g(s)\}, \end{aligned} \tag{2.4}$$

(L(U) stands for lower (upper) boundary) and let[4]

$$q_t(x, y) = \frac{\mathrm{d}}{\mathrm{d}y} \mathbf{P}^x\{W(t) \leqslant y,\ \sigma_{\mathrm{L}} \wedge \sigma_{\mathrm{U}} > t\}. \tag{2.5}$$

We take for q_t a version which is continuous in y on $(f(t), g(t))$. Such a version exists by the arguments of [6, Section 4.5–4.9]. Similarly, there exists a continuous function $\tilde{q}_t(y, x)$ of x on $(f(0), y(0))$ such that $\tilde{q}_t(y, x) = (\mathrm{d}/\mathrm{d}x)\mathbf{P}^y\{W(t) \leqslant x,\ f(t-s) < W(s) < g(t-s),\ 0 \leqslant s \leqslant t\}$ and as in [6, Section 4.8] we have

$$q_t(x, y) = \tilde{q}_t(y, x), \quad (x, y) \in (f(0), g(0)) \times (f(t), g(t)). \tag{2.6}$$

Lemma 2.1. *If*

$$|f(s)-f(0)| + |g(s)-g(0)| \leqslant A\{g(0)-f(0)\}^{-1}s \quad \textit{for } 0 \leqslant s \leqslant (4A)^{-1}\{g(0)-f(0)\}^2 \tag{2.7}$$

for some $A<\infty$, then there exists a constant $K_1 = K_1(A, \sigma) > 0$ such that

$$q_t(x_1, y)\,\mathrm{e}^{(\mu/\sigma^2)(y-x_1)} \geqslant K_1 \frac{x_1 - f(0)}{g(0)-f(0)} q_t(x_2, y)\,\mathrm{e}^{(\mu/\sigma^2)(y-x_2)} \tag{2.8}$$

for all $(4A)^{-1}\{g(0)-f(0)\}^2 \leqslant t \leqslant T$, $f(0) < x_1 \leqslant x_2 < g(0)$ and $f(t) < y < g(t)$.

Proof. We divide the proof in three steps. First we show that we may assume $\mu = 0$. Next we show that (2.8) in case $\mu = 0$ is implied by

$$\mathbf{P}^{x_1}\{\sigma_{\mathrm{U}} < \sigma_{\mathrm{L}} \wedge \Delta^2\} \geqslant K_1\{x_1 - f(0)\}\{g(0)-f(0)\}^{-1} \tag{2.9}$$

where

$$\Delta = \tfrac{1}{2}A^{-1/2}\{g(0)-f(0)\}. \tag{2.10}$$

Lastly we prove (2.9).

[3] As usual we take the inf of the empty set equal to $+\infty$ in the definition of all stopping times.
[4] $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$.

Step 1. Let $W^0(t)=W(t)+\mu t$. Then $W^0(t)$ is a Brownian motion with zero drift and variance coefficient σ^2. Since $W(t)=W^0(t)-\mu t$, the Cameron–Martin formula [11, Section 3.7] shows, for any Borel set B

$$\mathbf{P}^x\{W(t)\in B, \sigma_L\wedge\sigma_U>t\}=$$

$$=\int_B \mathbf{P}^x\{W^0(t)\in dy, \sigma_L\wedge\sigma_U>t\}\,e^{(1/2)(\mu/\sigma)^2t-(\mu/\sigma^2)(y-x)}.$$

Thus, if $q_t^0(x, y)$ is defined as q_t in (2.5) with W replaced by W^0, then

$$q_t(x,y)=\exp-\left\{\frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2t+\frac{\mu}{\sigma^2}(y-x)\right\}q_t^0(x,y). \tag{2.11}$$

It is immediate from (2.11) that it suffices to prove (2.8) for q_t^0, i.e. for the case $\mu=0$. *For the remainder of this proof we take* $\mu=0$ but we shall not indicate this in the notation by a superscript zero.

Step 2. For $x_1\leqslant x_2$ we shall construct two processes W_1, W_2 on the probability space $\tilde{\Omega}\equiv\Omega\times\Omega$ such that W_i is governed by the measure $\mathbf{P}^{x_i}$, $i=1,2$ and such that $W_1(t)=W_2(t)$ from some stopping time on. This so called coupling is achieved as follows: Let $\tilde{\mathbf{P}}$ be the product measure $\mathbf{P}^{x_1}\times\mathbf{P}^{x_2}$ on $\tilde{\Omega}$, and for a generic point $\omega=(\omega_1,\omega_2)$ of $\tilde{\Omega}$, let $W_0(s)=\omega_1(s)$, $W_2(s)=\omega_2(s)$, $s\geqslant 0$,

$$\tau=\inf\{s\geqslant 0: W_0(s)=W_2(s)\},$$

and finally

$$W_1(s)=\begin{cases}W_0(s), & s<\tau,\\ W_2(s), & s\geqslant\tau.\end{cases} \tag{2.12}$$

Clearly $W_2(\cdot)$ is governed by $\mathbf{P}^{x_2}$ and $W_0(\cdot)$ by $\mathbf{P}^{x_1}$. If we write $\tilde{\mathscr{F}}_s=\sigma$-field generated by $\{W_0(u), W_2(u): 0\leqslant u\leqslant s\}$, then τ is an $\{\tilde{\mathscr{F}}_s\}$ stopping time and by (2.12) and the strong Markov property, for any Borel sets B_i

$$\begin{aligned}&\tilde{\mathbf{P}}\{W_1(\tau+t_i)-W_1(\tau)\in B_i\mid 1\leqslant i\leqslant n|\tilde{\mathscr{F}}_\tau\}\\ &=\tilde{\mathbf{P}}\{W_2(\tau+t_i)-W_2(\tau)\in B_i\mid 1\leqslant i\leqslant n|\tilde{\mathscr{F}}_\tau\}\\ &=\mathbf{P}^z\{W(t_i)-W(0)\in B_i\mid 1\leqslant i\leqslant n\}\text{ on }W_2(\tau)=z\\ &=\mathbf{P}^{x_1}\{W(t_i)-W(0)\in B_i\mid 1\leqslant i\leqslant n\}.\end{aligned}$$

It is easy to see from this that W_1 is governed by $\mathbf{P}^{x_1}$. Now define σ_L^i, σ_U^i by replacing W by W_i in (2.4), $i=0,1,2$. Then

$$\begin{aligned}\mathbf{P}^{x_1}\{W(t)\in B\mid\sigma_L\wedge\sigma_U>t\}&=\tilde{\mathbf{P}}\{W_1(t)\in B\mid\sigma_L^1\wedge\sigma_U^1>t\}\\ &\geqslant\tilde{\mathbf{P}}\{W_2(t)\in B\mid\sigma_L^2\wedge\sigma_U^2>t\text{ and }\tau<\sigma_L^0\wedge\sigma_U^0\wedge t\}\\ &\geqslant\tilde{\mathbf{P}}\{W_2(t)\in B\mid\sigma_L^2\wedge\sigma_U^2>t\text{ and }\sigma_U^0<\sigma_L^0\wedge t\}.\end{aligned} \tag{2.13}$$

The last inequality in (2.13) follows from the intermediate value theorem, because $W_0(0)=x_1\leqslant x_2=W_2(0)$ a.e. [$\tilde{\mathbf{P}}$], and $W_0(\sigma_U^0)=g(\sigma_U^0)>W_2(\sigma_U^0)$ on the set $\{\sigma_U^0<\sigma_U^2\}\supset\{\sigma_L^2\wedge\sigma_U^2>t>\sigma_U^0\}$. Thus on the last set there exists an $s<\sigma_U^0$ such that $W_0(s)=W_2(s)$ and a fortiori $\tau<\sigma_U^0$. Finally $\sigma_L^0\wedge\sigma_U^0=\sigma_U^0$ whenever $\sigma_U^0<\sigma_L^0$. Since W_2 and W_0 are independent under $\tilde{\mathbf{P}}$ we obtain from (2.13)

$$\begin{aligned}\mathbf{P}^{x_1}\{W(t)\in B,\ \sigma_L\wedge\sigma_U>t\}&\geqslant\tilde{\mathbf{P}}\{W_2(t)\in B,\ \sigma_L^2\wedge\sigma_U^2>t\}\tilde{\mathbf{P}}\{\sigma_U^0<\sigma_L^0\wedge t\}\\&=\mathbf{P}^{x_2}\{W_2(t)\in B,\ \sigma_L^2\wedge\sigma_U^2>t\}\mathbf{P}^{x_1}\{\sigma_U<\sigma_L\wedge t\},\end{aligned}$$

and by differentiation

$$\begin{aligned}q_t(x_1,y)&\geqslant q_t(x_2,y)\mathbf{P}^{x_1}\{\sigma_U<\sigma_L\wedge t\}\\&\geqslant q_t(x_2,y)\mathbf{P}^{x_1}\{\sigma_U<\sigma_L\wedge\Delta^2\}\end{aligned}$$

for all $t\geqslant\Delta^2$. Thus, it indeed suffices to prove (2.9) when $\mu=0$. This will be done in

Step 3. Consider the linear functions

$$\mathrm{L}(s)=f(0)+A\{g(0)-f(0)\}^{-1}s$$

and

$$\mathrm{U}(s)=g(0)+A\{g(0)-f(0)\}^{-1}s,$$

and the stopping times

$$\begin{aligned}\tau_L&=\inf\{s\geqslant0: W(s)=\mathrm{L}(s)\},\\\tau_U&=\inf\{s\geqslant0: W(s)=\mathrm{U}(s)\}.\end{aligned}$$

Observe that by virtue of (2.7)

$$f(s)\leqslant\mathrm{L}(s)\leqslant g(s)\leqslant\mathrm{U}(s)\quad\text{for } 0\leqslant s\leqslant\Delta^2,$$

so that $\tau_U<\tau_L\wedge\Delta^2$ implies

$$\sigma_U\leqslant\tau_U<\tau_L\wedge\Delta^2\leqslant\sigma_L\wedge\Delta^2$$

(again use the intermediate value theorem). Thus,

$$\begin{aligned}\mathbf{P}^{x_1}\{\sigma_U<\sigma_L\wedge\Delta^2\}&\geqslant\mathbf{P}^{x_1}\{\tau_U<\tau_L\wedge\Delta^2\}\\&\geqslant\int_{y\geqslant\mathrm{U}(\Delta^2)}\mathbf{P}^{x_1}\{W(\Delta^2)\in\mathrm{d}y\}\mathbf{P}^{x_1}\{W(s)>\mathrm{L}(s),\ 0\leqslant s\leqslant\Delta^2\mid W(\Delta^2)=y\}.\end{aligned}\tag{2.14}$$

By (2.2) we have for $y\geqslant\mathrm{U}(\Delta^2)$

$$\begin{aligned}&\mathbf{P}^{x_1}\{W(s)>\mathrm{L}(s),\ 0\leqslant s\leqslant\Delta^2\mid W(\Delta^2)=y\}\\&\quad=1-\exp-2(\Delta\sigma)^2[x_1-f(0)][y-f(0)-A\{g(0)-f(0)\}^{-1}\Delta^2]\\&\quad\geqslant1-\exp-2(\Delta\sigma)^2[x_1-f(0)][\mathrm{U}(\Delta^2)-\mathrm{L}(\Delta^2)].\end{aligned}\tag{2.15}$$

Since

$$0<x_1-f(0)<g(0)-f(0)=2A^{1/2}\Delta$$

and

$$\mathrm{U}(\Delta^2)-\mathrm{L}(\Delta^2)=g(0)-f(0)=2A^{1/2}\Delta,$$

there exists a constant $K_2=K_2(A,\sigma)>0$ such that the last member of (2.15) is at least

$$K_2[x_1-f(0)]\Delta^{-1}=2K_2A^{1/2}\frac{x_1-f(0)}{g(0)-f(0)}.$$

(2.14) now yields

$$\begin{aligned}\mathbf{P}^{x_1}\{\sigma_{\mathrm{U}}<\sigma_{\mathrm{L}}\wedge\Delta^2\}&\geq\mathbf{P}^{x_1}\{W(\Delta^2)\geq\mathrm{U}(\Delta^2)\}2K_2A^{1/2}\frac{x_1-f(0)}{g(0)-f(0)}\\&=2K_2A^{1/2}\frac{x_1-f(0)}{g(0)-f(0)}[1-\Phi((\sigma\Delta)^{-1}[g(0)-x_1+A\{g(0)-f(0)\}^{-1}\Delta^2])]\\&\geq2K_2A^{1/2}\frac{x_1-f(0)}{g(0)-f(0)}[1-\Phi(\tfrac{5}{2}\sigma^{-1}A^{1/2})],\end{aligned}$$

where as usual

$$\Phi(v)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{v}\mathrm{e}^{-s^2/2}\,\mathrm{d}s.$$

(2.9) is now immediate with

$$K_1=2K_2A^{1/2}[1-\Phi(\tfrac{5}{2}\sigma^{-1}A^{1/2})].$$

For the formulation of the next proposition we introduce for $0\leq s\leq T$, $f(s)\leq x\leq g(s)$

$$\rho(x,s)=\min\{(x-f(s),g(s)-x\}. \tag{2.16}$$

Proposition 2.2. *Assume that for some $A<\infty$*

$$|f(s_2)-f(s_1)|+|g(s_2)-g(s_1)|\leq A\{g(s_1)-f(s_1)\}^{-1}(s_2-s_1) \tag{2.17}$$

for all $0\leq s_2-s_1\leq(4A)^{-1}\{g(s_1)-f(s_1)\}^2$, $0\leq s_1\leq s_2\leq T$. Then there exists a constant $K_3=K_3(A,\sigma)>0$ such that

$$K_3\leq\frac{q_t(x_1,y_1)\rho(x_2,0)\rho(y_2,t)}{q_t(x_2,y_2)\rho(x_1,0)\rho(y_1,t)}\exp\frac{\mu}{\sigma^2}\{(y_1-x_1)-(y_2-x_2)\}\leq K_3^{-1}$$

for all $(4A)^{-1}\{g(0)-f(0)\}^2\leq t\leq T$, $x_1,x_2\in(f(0),g(0))$ and $y_1,y_2\in(f(t),g(t))$.

Proof. As in step 1 of Lemma 2.1 (see (2.11) we may and shall assume that $\mu=0$ throughout this proof. By Lemma 2.1 we know that for all $f(0)<x_1\leq x_2<g(0)$ and $y\in(f(t), g(t))$

$$q_t(x_1, y)\geq K_1\frac{x_1-f(0)}{g(0)-f(0)}q_t(x_2, y)$$

and by interchanging the role of the upper and lower boundaries we only have to replace $x_1-f(0)$ by $g(0)-x_1$ in this inequality when $x_1>x_2$. Thus, for all $x_1, x_2\in(f(0), g(0))$

$$q_t(x_1, y)\geq K_1\frac{\rho(x_1, 0)}{g(0)-f(0)}q_t(x_2, y),$$

and therefore

$$q_t(x_1, y)\geq K_1\frac{\rho(x_1, 0)}{g(0)-f(0)}\sup_z q_t(z, y). \tag{2.18}$$

In particular, for

$$f(0)+\tfrac{1}{4}\{g(0)-f(0)\}\leq x_2\leq f(0)+\tfrac{3}{4}\{g(0)-f(0)\}=g(0)-\tfrac{1}{4}\{g(0)-f(0)\}$$

we have

$$q_t(x_2, y)\geq\tfrac{1}{4}K_1\sup_z q_t(z, y). \tag{2.19}$$

To derive an upper bound for $q_t(x_1, y)$ we introduce for $0\leq s<t\leq T$

$$r(x, y; s, t)=\frac{\mathrm{d}}{\mathrm{d}y}\mathbf{P}^x\{W(t-s)\leq y,\ f(s+u)<W(u)<g(s+u)\quad\text{for } 0\leq u\leq t-s\}. \tag{2.20}$$

Clearly $q_t(x, y)=r(x, y; 0, t)$ and for general s, $r(x, y; s, t)$ is a density of the same form as $q_{t-s}(x, y)$ except that $f(\cdot)$ and $g(\cdot)$ have been replaced by $f(s+\cdot)$ and $g(s+\cdot)$ respectively. Consequently we can again choose r such that it is continuous in y on $(f(t), g(t))$. We shall make this choice for r and apply (2.19) to r. By replacing A by $4A$ (and hence $K_1(A,\sigma)$ by $K_1(4A,\sigma)$) this yields

$$r(x_1, y; s, t)\geq\tfrac{1}{4}K_1\sup_z r(z, y; s, t) \tag{2.21}$$

for

$$(16A)^{-1}\{g(s)-f(s)\}^2\leq t-s,\quad 0\leq s\leq t\leq T,\quad\text{and}$$
$$f(s)+\tfrac{1}{4}\{g(s)-f(s)\}\leq x_1\leq g(s)-\tfrac{1}{4}\{g(s)-f(s)\},\quad y\in(f(t), g(t)). \tag{2.22}$$

In addition the Markov property implies

$$q_t(x, y) = \int_{f(s)}^{g(s)} q_s(x, z) r(z, y; s, t)\,\mathrm{d}z, \quad y \in (f(t), g(t)), \tag{2.23}$$

(compare (4.13) in [6]). Clearly (2.23) implies

$$\begin{aligned} q_t(x, y) &\leq \int_{f(s)}^{g(s)} q_s(x, z)\,\mathrm{d}z \sup_w r(w, y; s, t) \\ &= \mathbf{P}^x\{f(u) < W(u) < g(u), 0 \leq u \leq s\} \sup_w r(w, y; s, t). \end{aligned} \tag{2.24}$$

Take Δ as in (2.10) and $s = \frac{1}{2}\Delta^2$ and set

$$L^*(u) = f(0) - A\{g(0) - f(0)\}^{-1} u. \tag{2.25}$$

Then $f(u) \geq L^*(u)$, $0 \leq u \leq s$, and again applying (2.2), we have for some $K_4 = K_4(A, \sigma)$, $K_5 = K_5(A, \sigma) < \infty$

$$\begin{aligned} &\mathbf{P}^x\{f(u) < W(u) < g(u), 0 \leq u \leq s\} \leq \mathbf{P}^x\{W(u) > L^*(u), 0 \leq u \leq s\} \\ &= \int_{y > L^*(s)} \mathbf{P}^x\{W(s) \in \mathrm{d}y\} \mathbf{P}^x\{W(u) > L^*(u), 0 \leq u \leq s \mid W(s) = y\} \\ &= \frac{1}{\sigma\sqrt{2\pi s}} \int_{L^*(s)}^{\infty} \mathrm{e}^{-(2s\sigma^2)^{-1}(y-x)^2}\{1 - \exp - 2\sigma^{-2}s^{-1}[x - f(0)][y - L^*(s)]\}\,\mathrm{d}y \\ &\leq \frac{1}{\sigma\sqrt{2\pi s}} \int_{L^*(s)}^{\infty} \mathrm{e}^{-(2s\sigma^2)^{-1}(y-x)^2} 2\sigma^{-2}s^{-1}[x - f(0)][y - L^*(s)]\,\mathrm{d}y \\ &\leq K_4[x - f(0)]\left\{\frac{1}{s}(x - L^*(s)) + \frac{1}{\sigma\sqrt{2\pi s^3}} \int_{-\infty}^{+\infty} \mathrm{e}^{-(2s\sigma^2)^{-1}(y-x)^2} |y - x|\,\mathrm{d}y\right\} \\ &\leq K_5 \frac{x - f(0)}{g(0) - f(0)}. \end{aligned} \tag{2.26}$$

Substitution of (2.26) in (2.24) yields

$$q_t(x, y) \leq K_5 \frac{\rho(x, 0)}{g(0) - f(0)} \sup_z r(z, y; s, t) \tag{2.27}$$

when $f(0) < x \leq \frac{1}{2}(f(0) + g(0))$, $s = \frac{1}{2}\Delta^2$. This inequality remains valid for $\frac{1}{2}(f(0) + g(0)) < x < g(0)$ by interchange of the upper and lower boundaries.

On the other hand, still with $s = \frac{1}{2}\Delta^2$ we have

$$\begin{aligned} &|g(s) - g(0)| + |f(s) - f(0)| \leq A\{g(0) - f(0)\}^{-1}\tfrac{1}{2}\Delta^2 = \tfrac{1}{8}\{g(0) - f(0)\}, \\ &\tfrac{7}{8}\{g(0) - f(0)\} \leq g(s) - f(s) \leq \tfrac{9}{8}\{g(0) - f(0)\}, \\ &I \equiv [f(0) + \tfrac{7}{16}\{g(0) - f(0)\}, g(0) - \tfrac{7}{16}\{g(0) - f(0)\}] \\ &\quad \subset [f(s) + \tfrac{1}{4}\{g(s) - f(s)\}, g(s) - \tfrac{1}{4}\{g(s) - f(s)\}]. \end{aligned} \tag{2.28}$$

For $t \geq \Delta^2$ we have in addition

$$t-s \geq \tfrac{1}{2}\Delta^2 = (8A)^{-1}\{g(0)-f(0)\}^2 \geq (16A)^{-1}\{g(s)-f(s)\}^2.$$

Thus, by (2.21) and (2.22) we have for any $z \in I$

$$r(z, y; s, t) \geq \tfrac{1}{4}K_1 \sup_w r(w, y; s, t).$$

This together with (2.23) shows that

$$\begin{aligned} q_t(x, y) &\geq \int_I q_s(x, z)\,dz\, \tfrac{1}{4}K_1 \sup_w r(w, y; s, t) \\ &= \tfrac{1}{4}K_1 \sup_w r(w, y; s, t) \\ &\quad \cdot \mathbf{P}^x\{W(\tfrac{1}{2}\Delta^2) \in I, f(u) < W(u) < g(u), 0 \leq u \leq \tfrac{1}{2}\Delta^2\}. \end{aligned} \tag{2.29}$$

As in (2.28)

$$g(u) \geq g(0) - \tfrac{1}{8}\{g(0)-f(0)\}, \qquad f(u) \leq f(0) + \tfrac{1}{8}\{g(0)-f(0)\}, \quad 0 \leq u \leq \tfrac{1}{2}\Delta^2,$$

so that for $x \in I$

$$\begin{aligned} &\mathbf{P}^x\{W(\tfrac{1}{2}\Delta^2) \in I, f(u) < W(u) < g(u), 0 \leq u \leq \tfrac{1}{2}\Delta^2\} \geq \\ &\geq \mathbf{P}^0\{W(\tfrac{1}{2}\Delta^2) \in I - x, -\tfrac{5}{16}\{g(0)-f(0)\} \leq W(u) \leq \tfrac{5}{16}\{g(0)-f(0)\}, \\ &\qquad 0 \leq u \leq \tfrac{1}{2}\Delta^2\}. \end{aligned} \tag{2.30}$$

Since $I-x$ has length $\tfrac{7}{8}\{g(0)-f(0)\} = \tfrac{7}{4}A^{1/2}\Delta$ and contains the origin, $I-x$ covers at least one half of $(-\tfrac{5}{16}\{g(0)-f(0)\}, \tfrac{5}{16}\{g(0)-f(0)\})$. Standard estimates for Brownian motion (see [8, Problem 1.7.8], compare also Lemma 2.4 below) now show that the right hand side of (2.30) is at least $K_6 > 0$ for some $K_6 = K_6(A, \sigma)$ which depends on A and σ only. Thus, from (2.29)

$$\sup_z q_t(z, y) \geq \sup_{x \in I} q_t(x, y) \geq \tfrac{1}{4}K_1K_6 \sup_z r(z, y; s, t). \tag{2.31}$$

(2.27) and (2.31) together show

$$q_t(x, y) \leq K_7 \frac{\rho(x, 0)}{g(0)-f(0)} \sup_z q_t(z, y) \tag{2.32}$$

for some $K_7 = K_7(A, \sigma) < \infty$. Lastly, (2.18) and (2.32) give us for all $f(0) < x_1, x_2 < g(0)$, $f(t) < y < g(t)$

$$K_7^{-1}K_1 \leq \frac{q_t(x_1, y)\rho(x_2, 0)}{q_t(x_2, y)\rho(x_1, 0)} \leq K_7K_1^{-1}. \tag{2.33}$$

Thus, we have proved the proposition in the special case where $y_1 = y_2$.

To handle ratios of q_t with different second arguments we use the relation (see (2.6))

$$\frac{q_t(x, y_1)}{q_t(x, y_2)} = \frac{\tilde{q}_t(y_1, x)}{\tilde{q}_t(y_2, x)}.$$

The functions $s \to f(t-s)$ and $s \to g(t-s)$, $0 \leq s \leq t$, satisfy condition (2.17) with A replaced by $\frac{3}{2}A$. Indeed, if $0 \leq s_2 - s_1 \leq (4A)^{-1}\{g(s_1) - f(s_1)\}^2$, then, just as in (2.28), $g(s_2) - f(s_2) \leq \frac{3}{2}\{g(s_1) - f(s_1)\}$, and hence, by (2.17)

$$|f(s_1) - f(s_2)| + |g(s_1) - g(s_2)| \leq \tfrac{3}{2}A\{g(s_2) - f(s_2)\}^{-1}(s_2 - s_1).$$

Thus, short of replacing $K_1(A, \sigma)$ and $K_7(A, \sigma)$ by $K_1(\frac{3}{2}A, \sigma)$ and $K_7(\frac{3}{2}A, \sigma)$ we may apply (2.33) to $\tilde{q}_t$ to obtain

$$K_7^{-1}K_1 \leq \frac{\tilde{q}_t(y_1, x)\rho(y_2, t)}{\tilde{q}_t(y_2, x)\rho(y_1, t)} \leq K_7K^{-1}. \tag{2.34}$$

The proposition now follows with $K_3 = K_7^{-2}K_1^2$ by combining (2.33) and (2.34).

Corollary 2.3. *Assume the hypotheses of Proposition* 2.2 *hold. Then* (*see* (2.20) *for definition of* r)

$$\begin{aligned} K_3 &\leq \frac{r(x_1, y_1; s, t)\rho(x_2, s)\rho(y_2, t)}{r(x_2, y_2; s, t)\rho(x_1, s)\rho(y_1, t)} \exp \frac{\mu}{\sigma^2}\{(y_1 - x_1) - (y_2 - x_2)\} \\ &\leq K_3^{-1} \end{aligned} \tag{2.35}$$

whenever

$$(4A)^{-1}\{g(s) - f(s)\}^2 \leq t - s, \quad 0 \leq s \leq t \leq T, \tag{2.36}$$

and $x_1, x_2 \in (f(s), g(s))$, $y_1, y_2 \in (f(t), g(t))$. *Also, if* W^0 *is a Brownian motion with mean zero and variance coefficient* σ^2, *then there exists a* $K_8 = K_8(A, \sigma) > 0$ *such that*

$$\begin{aligned} K_8 \frac{\rho(x, s)}{g(s) - f(s)} &\leq \Big[\sup_{f(s) < z < g(s)} \mathbf{P}^z\{f(s+u) < W^0(u) < g(s+u)\, 0 \leq u \leq t - s\}\Big]^{-1} \\ &\quad \cdot \mathbf{P}^x\{f(s+u) < W^0(u) < g(s+u), \quad 0 \leq u \leq t - s\} \\ &\leq K_8^{-1} \frac{\rho(x, s)}{g(s) - f(s)} \end{aligned} \tag{2.37}$$

for, s, t satisfying (2.36) *and* $f(s) < x < g(s)$. *Also, if* $0 < \varepsilon_1, \varepsilon_2 < \frac{1}{2}$, *then there exists a* $K_9 = K_9(A, \sigma, \varepsilon_1, \varepsilon_2) > 0$ *such that for all* $t \geq (4A)^{-1}\{g(0) - f(0)\}^2$ *and*

$$\begin{aligned} &x \in [f(0) + \varepsilon_1(g(0) - f(0)), g(0) - \varepsilon_1(g(0) - f(0))], \\ &\mathbf{P}^x\{f(u) < W^0(u) < g(u), 0 \leq u \leq t, \\ &\quad f(t) + \varepsilon_2(g(t) - f(t)) < W^0(t) < g(t) - \varepsilon_2(g(t) - f(t))\} \geq \\ &\quad \geq K_9 \sup_z \mathbf{P}^z\{f(u) < W^0(u) < g(u): 0 \leq u \leq t\}. \end{aligned} \tag{2.38}$$

Lastly, if $(4A)^{-1}\{g(0) - f(0)\}^2 \leq s \leq T$ *and* $(4A)^{-1}\{g(s) - f(s)\}^2 \leq t - s$, $t \leq T$ *and* $g(0) - f(0) \geq 2$, $g(s) - f(s) \geq 2$, *then*

$$\begin{aligned} &\mathbf{P}^1\{f(u) < W^0(u) < g(u), 0 \leq u \leq t\} \geq \\ &\quad \geq K_8(g(s) - f(s))\mathbf{P}^1\{f(u) < W^0(u) < g(u), 0 \leq u \leq s\} \\ &\quad\quad \cdot \mathbf{P}^1\{f(s+u) < W^0(u) < f(s+u), 0 \leq u \leq t - s\}. \end{aligned} \tag{2.39}$$

Proof. (2.35) is merely Proposition 2.2 applied to the functions $u \to f(s+u)$ and $u \to g(s+u)$, $0 \leq u \leq T - s$. (2.37) is immediate from (2.35) with $\mu = 0$. Also (2.38) follows easily directly from Proposition 2.2 with $\mu = 0$. Indeed, the left hand side of (2.38) equals

$$\int^{(t)} q_t(x, y)\,\mathrm{d}y$$

where $\int^{(t)}$ runs over $(f(t) + \varepsilon_2(g(t) - f(t)), g(t) - \varepsilon_2(g(t) - f(t)))$. We now observe that for y in this interval $\rho(y, t) \geq \varepsilon_2(g(t) - f(t)) = 2\varepsilon_2 \sup_z \rho(z, t)$ and that similarly, for all $f(0) + \varepsilon_1(g(0) - f(0)) \leq x \leq g(0) + \varepsilon_1(g(0) - f(0))$ one has $\rho(x, 0) \geq 2\varepsilon_1 \sup_z \rho(z, 0)$.

As for (2.39), this follows from

$$\begin{aligned} &\mathbf{P}^1\{f(u) < W^0(u) < g(u), 0 \leq u \leq t\} \geq \\ &\quad \geq \int_{\rho(y,s) \geq \frac{1}{4}(g(s) - f(s))} q_s(1, y)\mathbf{P}^y\{f(s+u) < W^0(u) < g(s+u), 0 \leq u \leq t - s\}\,\mathrm{d}y \end{aligned}$$

together with

$$\begin{aligned} &\mathbf{P}^y\{f(s+u) < W^0(u) < g(s+u), 0 \leq u \leq t - s\} = \int r(y, z, s, t)\,\mathrm{d}z \\ &\quad \geq K_3 \frac{\rho(y, s)}{\rho(1, s)} \int r(1, z, s, t)\,\mathrm{d}z \\ &\quad \geq \tfrac{1}{4}K_3(g(s) - f(s))\mathbf{P}^1\{f(s+u) < W^0(u) < g(s+u), 0 \leq u \leq t - s\} \end{aligned}$$

(if $\rho(y, s) \geq \frac{1}{4}\{g(s) - f(s)\}$), and the argument used for (2.38).

Lemma 2.4. *Let W^0 be a Brownian motion with mean zero and variance coefficient σ^2. Then for $t>0$, $\Delta>0$ and $0<x, y<\Delta$,*

$$\mathbf{P}^x\{0<W^0(u)<\Delta: 0\le u\le t,\ W^0(t)\in \mathrm{d}y\} \tag{2.40}$$

has a density $s_t(x, y; \Delta)$ which satisfies

$$\left[1-5\left\{1-\exp-\frac{t\sigma^2\pi^2}{2\Delta^2}\right\}^{-2}\exp-\frac{3t\sigma^2\pi^2}{2\Delta^2}\right]$$
$$\cdot\frac{2}{\Delta}\sin\frac{\pi x}{\Delta}\sin\frac{\pi y}{\Delta}\exp-\frac{t\sigma^2\pi^2}{2\Delta^2}$$
$$\le s_t(x, y; \Delta)$$
$$\le\left[1+5\left\{1-\exp-\frac{t\sigma^2\pi^2}{2\Delta^2}\right\}^{-2}\exp-\frac{3t\sigma^2\pi^2}{2\Delta^2}\right]$$
$$\cdot\frac{2}{\Delta}\sin\frac{\pi x}{\Delta}\sin\frac{\pi y}{\Delta}\exp-\frac{t\sigma^2\pi^2}{2\Delta^2}.$$

Proof. By [8, Problem 1.7.8], (2.40) has the density

$$s_t(x, y; \Delta)=\frac{2}{\Delta}\sum_{n=1}^{\infty}\exp\{-n^2t\sigma^2\pi^2/2\Delta^2\}\sin\frac{\pi xn}{\Delta}\sin\frac{\pi yn}{\Delta}.$$

One easily shows that

$$\left|\sin\frac{\pi xn}{\Delta}\right|\le n\sin\frac{\pi x}{\Delta},\quad 0\le x\le\Delta,$$

and the same inequality with x replaced by y. Consequently, for some $|\theta|\le 1$

$$s_t(x, y; \Delta)=\frac{2}{\Delta}\exp\{-t\sigma^2\pi^2/2\Delta^2\}\sin\frac{\pi x}{\Delta}\sin\frac{\pi y}{\Delta}$$
$$\cdot\left[1+\theta\sum_{n=2}^{\infty}\exp\{-(n^2-1)(t\sigma^2\pi^2)/2\Delta^2\}n^2\right].$$

The lemma now follows from

$$\sum_{n=2}^{\infty}\exp\{-n^2(t\sigma^2\pi^2)/2\Delta^2\}n^2\le\sum_{k=4}^{\infty}k\exp\{-k(t\sigma^2\pi^2)/2\Delta^2\}$$
$$=-\left[\frac{\mathrm{d}}{\mathrm{d}u}\sum_{k=4}^{\infty}\mathrm{e}^{-ku}\right]_{u=t\sigma^2\pi^2/2\Delta^2}\le\left[\frac{5\,\mathrm{e}^{-4u}}{(1-\mathrm{e}^{-u})^2}\right]_{u=t\sigma^2\pi^2/2\Delta^2}.$$

Proposition 2.5. *Let f, g be two functions on $[0, T]$ which satisfy the following conditions: There exist two differentiable functions F, G and a decreasing function ε and constants $0<B_i<\infty$ such that on $[0, T]$*

$$|g(s)-G(s)|\le B_1,\qquad |f(s)-F(s)|\le B_1, \tag{2.41}$$

$$G(s)-F(s) \geq B_2+2B_1, \qquad \varepsilon(0) \leq B_3, \tag{2.42}$$

$$|F'(s)|+|G'(s)| \leq \varepsilon(s)\{(G(s)-F(s))\log(s+2)\}^{-1}. \tag{2.43}$$

$$\varepsilon(s) \leq \sigma^2 \pi B_2\{3200(B_2+2B_1)\}^{-1} \quad \text{for } s \geq B_4. \tag{2.44}$$

Again let W^0 be a Brownian motion with mean zero and variance coefficient σ^2. Then there exists a constant $K_0<\infty$, depending on σ and B_1-B_4 only, such that for all $0 \leq t \leq T$, and

$$x \in [F(0)+B_1+\tfrac{1}{4}(G(0)-F(0)-2B_1), G(0)-B_1-\tfrac{1}{4}(G(0)-F(0)-2B_1)],$$

$$\exp\Bigg[-\frac{\sigma^2\pi^2}{2}\int_0^t \frac{ds}{(g(s)-f(s))^2}-K_0\Bigg\{1+\int_0^t \frac{\varepsilon(s)\,ds}{(g(s)-f(s))^2}+\int_0^t \frac{B_1\,ds}{(g(s)-f(s))^3}\Bigg\}$$
$$-(K_0 \wedge \varepsilon(0))\log(t+2)\Bigg] \leq$$
$$\leq \mathbf{P}^x\{f(s)<W^0(s)<g(s), 0 \leq s \leq t\}$$
$$\leq \exp\Bigg[-\frac{\sigma^2\pi^2}{\pi}\int_0^t \frac{ds}{(g(s)-f(s))^2}+K_0\Bigg\{1+\int_0^t \frac{\varepsilon(s)\,ds}{(g(s)-f(s))^2}+\int_0^t \frac{B_1\,ds}{(g(s)-f(s))^3}\Bigg\}$$
$$+(K_0 \wedge \varepsilon(0))\log(t+2)\Bigg]. \tag{2.45}$$

Exactly the same bounds hold for

$$\mathbf{P}^x\{f(t-s)<W^0(s)<g(t-s), \quad 0 \leq s \leq t\} \tag{2.46}$$

whenever $0 \leq t \leq T$,

$$x \in [F(t)+B_1+\tfrac{1}{4}(G(t)-F(t)-2B_1), G(t)-B_1-\tfrac{1}{4}(G(t)-F(t)-2B_1)].$$

Remark 2.6. When $B_1=0$, i.e., when f and g themselves satisfy (2.42) and the smoothness condition (2.43), then f and g satisfy the hypotheses of Proposition 2.2 for any $A \geq 2\varepsilon(0)(\log 2)^{-1}$ (compare the argument given below to establish (2.51)). An application of (2.37) with $A=2B_3(\log 2)^{-1}$ therefore allows as in the case $B_1=0$ to sharpen (2.45) to

$$\sin\frac{\pi(x-f(0))}{g(0)-f(0)}\exp\Bigg[-\tfrac{1}{2}\sigma^2\pi^2\int_0^t \frac{ds}{(g(s)-f(s))^2}-K_0\Bigg\{1+\int_0^t \frac{\varepsilon(s)\,ds}{(g(s)-f(s))^2}\Bigg\}$$
$$-(K_0 \wedge \varepsilon(0))\log(t+2)\Bigg] \leq$$
$$\leq \mathbf{P}^x\{f(s)<W^0(s)<g(s), 0 \leq s \leq t\}$$
$$\leq \sin\frac{\pi(x-f(0))}{g(0)-f(0))}\exp\Bigg[-\tfrac{1}{2}\sigma^2\pi^2\int_0^t \frac{ds}{(g(s)-f(s))^2}+K_0\Bigg\{1+\int_0^t \frac{\varepsilon(s)\,ds}{(g(s)-f(s))^2}\Bigg\}$$
$$+(K_0 \wedge \varepsilon(0))\log(t+2)\Bigg], \tag{2.47}$$

whenever $(8B_3)^{-1}(\log 2)(g(0)-f(0))^2 \leq t \leq T$, $x \in (f(0), g(0))$. (Note that $\sin \pi(x - f(0))/(g(0)-f(0))$ is of the same order as $(g(0)-f(0))^{-1}\rho(x, 0)$). Also

$$\sin \frac{\pi(x-f(t))}{g(t)-f(t)} \exp\left[-\tfrac{1}{2}\sigma^2\pi^2 \int_0^t \frac{ds}{(g(s)-f(s))^2} - K_0\left\{1+\int_0^t \frac{\varepsilon(s)\,ds}{(g(s)-f(s))^2}\right\} - (K_0 \wedge \varepsilon(0))\log(t+2)\right] \leq$$

$$\leq \mathbf{P}^x\{f(t-s) < W^0(s) < g(t-s),\ 0 \leq s \leq t\}$$

$$\leq \sin \frac{\pi(x-f(t)}{g(t)-f(t)} \exp\left[-\tfrac{1}{2}\sigma^2\pi^2 \int_0^t \frac{ds}{(g(s)-f(s))^2} + K_0\left\{1+\int_0^t \frac{\varepsilon(s)\,ds}{(g(s)-f(s))^2}\right\} + (K_0 \wedge \varepsilon(0))\log(t+2)\right] \tag{2.48}$$

whenever $(8B_3)^{-1}(\log 2)(g(t)-f(t))^2 \leq t \leq T$, $x \in (f(t), g(t))$.

Remark 2.7. For the cases which interest us the behavior of the estimates in (2.45)–(2.48) is principally governed by

$$\exp -\tfrac{1}{2}\sigma^2\pi^2 \int_0^t \frac{ds}{(g(s)-f(s))^2}.$$

The other factors will usually play a smaller role. In Section 3 we shall in particular apply Proposition 2.5 with $f \equiv 0$, $g(s) = A(B+s)^\lambda$ for some $A, B > 0$, $0 < \lambda < \frac{1}{2}$. We can then take $B_1 = 0$, $\varepsilon(s) = \lambda A^2 (B+s)^{2\lambda-1}\log(s+2)$ for large enough s, and we find for $0 < x < AB^\lambda$ and $t \to \infty$

$$\mathbf{P}^x\{0 < W^0(s) < A(B+s)^\lambda : 0 \leq s \leq t\} =$$

$$= \sin(\pi x A^{-1}B^{-\lambda}) \exp\left[-\frac{\sigma^2\pi^2}{2(1-2\lambda)A^2}\{(B+t)^{1-2\lambda} - B^{1-2\lambda}\} + 0(\log t)^2\right],$$

where the constant in $0(\log t)^2$ depends on σ, A, B and λ only.

Proof. *Throughout this proof K_i will stand for constants which depend on σ, $B_1 - B_4$ only.* Practically the same proof as will be given for (2.45) can be used to estimate (2.46). Alternatively, we can derive one estimate from the other by the use of (2.6). We shall therefore restrict ourselves to (2.45). Without loss of generality we take $t = T$ and $\varepsilon(\cdot)$ right continuous (we can always replace $\varepsilon(s)$ by $\lim_{u \downarrow s} \varepsilon(u)$). The bounds in (2.45) are proved by approximating f and g from above and below by step functions and then by estimating the probability that W^0 stays between two step functions. Since the proof of the upper and lower bound in (2.45) are almost identical we prove the lower bound only.

Step 1. In this step we construct the step function approximations to f and g and give some preliminary estimates on the distance between f and g and their approximations. Set

$$h(s)=g(s)-f(s),\qquad H(s)=G(s)-F(s),\quad C=8(\sigma\pi)^{-2}$$

and construct an increasing sequence of points as follows:

$$a_0=4+\inf\{s\geq 0: \varepsilon(s)\leq(200\pi C)^{-1}B_2(2B_2+4B_1)^{-1}\} \tag{2.49}$$

($=T$ if no such s exists) and

$$a_{n+1}=a_n+CH^2(a_n)\log(a_n+2),\quad n\geq 0, \tag{2.50}$$

as long as $a_n<T$. If $a_0<T$, let N be the unique index for which $a_N<T\leq a_{N+1}$. If $N\geq 1$ we set $I_n=[a_n, a_{n+1})$, $0\leq n\leq N-2$, $I_{N-1}=[a_{N-1}, T]$. If $N=0$ we only introduce the one interval $I_0=[a_0, T]$. Before we come to the main estimates we need an estimate on the fluctuations of F, G over I_n and for the growth of H. Specifically, we shall show[5]

$$\begin{aligned}F(a_n)-\frac{5}{\sqrt{2}}C\varepsilon(a_n)H(a_n)&\leq F(s)\\ &\leq F(a_n)+\frac{5}{\sqrt{2}}C\varepsilon(a_n)H(a_n),\quad s\in\bar{I}_n,\ n\geq 0,\end{aligned} \tag{2.51}$$

and the same inequality with F replaced by G, and for some K_{10}

$$H^2(s)\leq H^2(a_0)+K_{10}(s+1)\{\log(s+2)\}^{-1},\quad s\geq a_0. \tag{2.52}$$

Because we assumed $\varepsilon(\cdot)$ decreasing

$$\varepsilon(s)\leq(200\pi C)^{-1}B_2(2B_2+4B_1)^{-1},\quad s\geq a_0. \tag{2.53}$$

We therefore have from (2.43) for $a_0<s\leq T$

$$|H'(s)|\leq|F'(s)|+|G'(s)|\leq\varepsilon(s)\{H(s)\log(s+2)\}^{-1},$$

whence, for $a_0\leq a_n<s\leq a_{n+1}\wedge T$,

$$\left|\frac{\mathrm{d}H^2(s)}{\mathrm{d}s}\right|\leq 2\varepsilon(a_n)\{\log(a_n+2)\}^{-1},$$

$$\begin{aligned}|H^2(s)-H^2(a_n)|&\leq 2\varepsilon(a_n)\{\log(a_n+2)\}^{-1}(a_{n+1}-a_n)\\ &\leq 2C\varepsilon(a_n)H^2(a_n)\leq\tfrac{1}{2}H(a_n).\end{aligned}$$

Thus, for $0\leq n\leq N-2$, $s\in\bar{I}_n$, or for $n=N$ and $a_n\leq s\leq T$,

$$\frac{1}{\sqrt{2}}H(a_n)\leq H(s)\leq\sqrt{\frac{3}{2}}H^2(a_n) \tag{2.54}$$

and for $s\in\bar{I}_{N-1}$

$$\tfrac{1}{2}H(a_{N-1})\leq H(s)\leq\tfrac{3}{2}H(a_{N-1}), \tag{2.55}$$

[5] $\bar{I}$ denotes the closure of I.

because for $a_N \leqslant s \leqslant T$ we have $H(s) \geqslant (1/\sqrt{2})H(a_N) \geqslant \frac{1}{2}H(a_{N-1})$ and similarly for the right hand inequality. As a consequence of (2.54) and (2.43) we have for $a_n \leqslant s \leqslant a_{n+1} \wedge T$

$$\begin{aligned}
&|F(s)-F(a_n)|+|G(s)-G(a_n)| \leqslant \\
&\quad \leqslant \varepsilon(a_n)\left\{\frac{1}{\sqrt{2}}H(a_n)\log(a_n+2)\right\}^{-1}(s-a_n) \\
&\quad \leqslant \varepsilon(a_n)\sqrt{2}CH(a_n).
\end{aligned} \tag{2.56}$$

If $n \leqslant N-2$ then (2.56) holds for all $s \in \bar{I}_n$, whereas, as before, for $s \in \bar{I}_{N-1}$

$$\begin{aligned}
&|F(s)-F(a_{N-1})|+|G(s)-G(a_{N-1})| \leqslant \\
&\quad \leqslant \varepsilon(a_{N-1})\sqrt{2}C(H(a_{N-1})+H(a_N)) \leqslant \varepsilon(a_{N-1})\frac{5}{\sqrt{2}}CH(a_{N-1}).
\end{aligned}$$

This proves (2.51). (2.52) is even easier. As above

$$\left|\frac{\mathrm{d}H^2(s)}{\mathrm{d}s}\right| \leqslant 2\varepsilon(a_0)\{\log(s+2)\}^{-1}, \quad s > a_0,$$

so that

$$\begin{aligned}
H^2(s) &\leqslant H^2(a_0)+2\varepsilon(a_0)\int_0^s \frac{\mathrm{d}u}{\log(u+2)} \\
&\leqslant H^2(a_0)+K_{11}\int_0^s \frac{\mathrm{d}u}{\log(u+2)} \quad \text{(see (2.53))} \\
&\leqslant H^2(a_0)+K_{10}\frac{s+1}{\log(s+2)},
\end{aligned}$$

which is just (2.52).

We now construct approximations to f and g as follows: Let

$$m_n = B_1 + \sup_{s\in I_n} F(s), \qquad M_n = -B_1 + \inf_{s\in I_n} G(s)$$

and set

$$\tilde{f}(s) = f(s)I_{[0,a_0)}(s) + \sum_0^{N-1} m_n I_n(s),$$

$$\tilde{g}(s) = g(s)I_{[0,a_0)}(s) + \sum_0^{N-1} M_n I_n(s),$$

where $I_n(\cdot)$ stands for the indicator function of the interval I_n. We shall also use the following abbreviations:

$$\lambda_n = \text{length of } I_n = \begin{cases} a_{n+1}-a_n & \text{if } n \leq N-2, \\ T-a_{N-1} & \text{if } n = N-1 \geq 0, \\ T-a_0 & \text{if } n = N = 0, \end{cases}$$

$$\Delta_n = M_n - m_n,$$

and if $a_0 < T$, then

$$\Lambda_0 = (f(a_0) \vee m_0, g(a_0) \wedge M_0),$$

$$\Lambda_{n+1} = (m_n \vee m_{n+1}, M_n \wedge M_{n+1}), \quad 0 \leq n \leq N-2,$$

$$\Lambda_N = (m_{N-1}, M_{N-1}) \quad \text{if } N \geq 1,$$

$$\Lambda_1 = (m_0, M_0) \quad \text{if } N = 0.$$

Then, by definition $\tilde{f}(s) = f(s)$ and $g(s) = \tilde{g}(s)$ for $s < a_0 \wedge T$, while for all s, $\tilde{f}(s) \geq f(s)$, $\tilde{g}(s) \leq g(s)$ (see (2.41)). Consequently, if $a_0 < T$,[6]

$$\begin{aligned} &\mathbf{P}^x\{f(s) < W^0(s) < g(s), 0 \leq s \leq T\} \geq \\ &\quad \geq \mathbf{P}^x\{\tilde{f}(s) < W^0(s) < \tilde{g}(s), 0 \leq s \leq T\} \\ &\quad = \int_{\Lambda_0} \mathbf{P}^x\{f(s) < W^0(s) < g(s), 0 \leq s < a_0, W^0(a_0) \in dy_0\} \\ &\quad \prod_0^{(N-1)^+} \int_{y_{n+1} \in \Lambda_{n+1}} \mathbf{P}^{y_n}\{m_n < W^0(s) < M_n, 0 \leq s < \lambda_n, W^0(\lambda_n) \in dy_{n+1}\}. \end{aligned} \tag{2.57}$$

Step 2. The remainder of the proof consists in estimating the various factors in (2.57). In this step we estimate the product of the integrals in the right hand side of (2.57), *and until further notice we assume* $a_0 < T$ *and* $N > 1$. By an application of Lemma 2.4 we then have

$$\begin{aligned} &\mathbf{P}^{y_n}\{m_n < W^0(s) < M_n, 0 \leq s < \lambda_n, W^0(\lambda_n) \in dy_{n+1}\} \geq \\ &\quad \geq \left[1 - 5\left\{1 - \exp -\frac{\lambda_n \sigma^2 \pi^2}{2\Delta_n^2}\right\}^{-2} \exp -\frac{3\lambda_n \sigma^2 \pi^2}{2\Delta_n^2}\right] \\ &\quad \cdot \frac{2}{\Delta_n} \sin \frac{\pi(y_n - m_n)}{\Delta_n} \sin \frac{\pi(y_{n+1} - m_n)}{\Delta_n} \exp -\frac{\lambda_n \sigma^2 \pi^2}{2\Delta_n^2} \, dy_{n+1}. \end{aligned} \tag{2.58}$$

Clearly

$$m_n \geq F(a_n) + B_1, M_n \leq G(a_n) - B_1, H(a_n) \geq M_n - m_n + 2B_1, \tag{2.59}$$

[6] $a^+ = \max(a, 0)$.

and by our choice of a_n and C

$$\lambda_n \geq a_{n+1} - a_n = CH^2(a_n)\log(a_n+2)$$
$$\geq C(M_n - m_n)^2 \log(a_n+2) \geq \frac{2}{\sigma^2\pi^2}\Delta_n^2 \log(a_n+2).$$

Consequently,

$$1 - 5\left\{1 - \exp\frac{-\lambda_n\sigma^2\pi^2}{2\Delta_n^2}\right\}^{-2} \exp -\frac{3\lambda_n\sigma^2\pi^2}{2\Delta_n^2} \geq$$
$$\geq 1 - 20\exp -3\log(a_n+2) \geq 1 - 20a_n^{-3} \geq \exp -40a_n^{-3}. \tag{2.60}$$

Next we deduce from (2.51)

$$m_n \leq F(a_n) + B_1 + \frac{5}{\sqrt{2}} C\varepsilon(a_n)H(a_n),$$

and from the analogue of (2.51) for G,

$$M_n \geq G(a_n) - B_1 - \frac{5}{\sqrt{2}} C\varepsilon(a_n)H(a_n).$$

Combined with (2.59), (2.42) and (2.53) this gives

$$B_2(2B_2+4B_1)^{-1}H(a_n) \leq H(a_n)\{1 - 5\sqrt{2}C\varepsilon(a_n)\} - 2B_1 \leq \Delta_n$$
$$\leq H(a_n) - 2B_1, \tag{2.61}$$

and using (2.51), (2.41) once more,

$$|\Delta_n - h(s)| \leq 2B_1 + |\Delta_n - H(a_n)| + |H(s) - H(a_n)|$$
$$\leq 4B_1 + 10\sqrt{2}C\varepsilon(a_n)H(a_n), \quad s \in \bar{I}_n, 0 \leq n < N. \tag{2.62}$$

Thus

$$\left|\frac{\lambda_n}{\Delta_n^2} - \int_{I_n} \frac{\mathrm{d}s}{h^2(s)}\right| \leq \int_{I_n} |\Delta_n^{-2} - h^{-2}(s)| \,\mathrm{d}s$$
$$\leq \int_{I_n} \Delta_n^{-2}h^{-2}(s)\{\Delta_n + h(s)\}|\Delta_n - h(s)| \,\mathrm{d}s$$
$$\leq 2\Delta_n^{-2} \int_{I_n} h^{-1}(s)\{4B_1 + 10\sqrt{2}C\varepsilon(a_n)H(a_n)\} \,\mathrm{d}s$$

(because $h(s) \geq \Delta_n$).

By (2.41), (2.51), (2.53)

$$h(s) \geq H(s) - 2B_1 \geq \{1 - 5\sqrt{2}C\varepsilon(a_n)\}H(a_n) - 2B_1$$
$$\geq \left\{1 - 5\sqrt{2}C\varepsilon(a_n) - \frac{2B_1}{B_2+2B_1}\right\}H(a_n) \geq \frac{B_2}{2B_2+4B_1}H(a_n), \quad s \in I_n,$$

so that for some K_{12} and $0 \leq n < N$

$$\left|\frac{\lambda_n}{\Delta_n^2} - \int_{I_n} \frac{\mathrm{d}s}{h^2(s)}\right| \leq K_{12}\left\{\frac{\lambda_n \varepsilon(a_n)}{\Delta_n^2} + \frac{B_1}{\Delta_n^2}\int_{I_n}\frac{\mathrm{d}s}{h(s)}\right\}. \tag{2.63}$$

Now for $n \leq N-2$

$$\lambda_n = CH^2(a_n)\log(a_n+2) \leq K_{13}\,\Delta_n^2 \log(a_n+2) \tag{2.64}$$

for some K_{13} (see (2.61)). Similarly, for $n = N-1$

$$\begin{aligned}\lambda_{N-1} &\leq C\{H^2(a_N)\log(a_N+2) + H^2(a_{N-1})\log(a_{N-1}+2)\}\\ &\leq CH^2(a_{N-1})\{\tfrac{3}{2}\log(a_N+2) + \log(a_{N-1}+2)\} \quad \text{(see (2.55))}.\end{aligned} \tag{2.65}$$

But, by (2.52) and $a_n \geq a_1 \geq CH^2(a_0)\log 2$, $n \geq 1$,

$$\begin{aligned}a_{n+1} - a_n = CH^2(a_n)\log(a_n+2) &\leq CH^2(a_0)\log(a_n+2) + CK_{10}(a_n+1)\\ &\leq (a_n+1)\{\log(a_n+2)(\log 2)^{-1} + CK_{10}\}, n \geq 1.\end{aligned}$$

Therefore

$$\log(a_{n+1}+2) \leq K_{14}\log(a_n+2), \quad n \geq 1. \tag{2.66}$$

Thus, by raising K_{13}, if necessary, (2.64) remains valid even for $n = N-1$ (recall that we are still considering only the case $N > 1$). (2.63)–(2.65) yield

$$\begin{aligned}&\left|\frac{\lambda_n}{\Delta_n^2} - \int_{I_n}\frac{\mathrm{d}s}{h^2(s)}\right| \leq\\ &\quad \leq K_{15}\left\{\varepsilon(a_n)\log(a_{n-1}+2) + \frac{B_1}{\Delta_n^2}\int_{I_n}\frac{\mathrm{d}s}{h(s)}\right\}, \quad 2 \leq n \leq N-1.\end{aligned}$$

Lastly, by (2.62) and (2.61)

$$1 \leq \frac{h(s)}{\Delta_n} \leq K_{16}, \quad s \in I_n, \tag{2.67}$$

(the left hand side inequality is trivial from (2.41)) so that

$$\begin{aligned}&\left|\frac{\lambda_n}{\Delta_n^2} - \int_{I_n}\frac{\mathrm{d}s}{h^2(s)}\right| \leq\\ &\quad \leq K_{17}\left\{\int_{I_{n-1}}\frac{\varepsilon(s)}{h^2(s)}\mathrm{d}s + B_1\int_{I_n}\frac{\mathrm{d}s}{h^3(s)}\right\}, \quad 2 \leq n \leq N-1.\end{aligned} \tag{2.68}$$

For $n = 0$ or $n = 1$ we obtain (see (2.63)–(2.65))

$$\begin{aligned}\left|\frac{\lambda_n}{\Delta_n^2} - \int_{I_n}\frac{\mathrm{d}s}{h^2(s)}\right| &\leq K_{18}\varepsilon(a_n)\log(T+2) + K_{17}B_1\int_{I_n}\frac{\mathrm{d}s}{h^3(s)}\Big\}\\ &\leq K_{18}\varepsilon(a_0)\log(T+2) + K_{17}B_1\int_{I_n}\frac{\mathrm{d}s}{h^3(s)}.\end{aligned} \tag{2.69}$$

By (2.53) $\varepsilon(a_0)$ is bounded by some K_{19} as well as by $\varepsilon(0)$ so that the last member of (2.69) is bounded by

$$(K_{19}\wedge\varepsilon(0))\log(T+2)+K_{17}B_1\int_{I_n}\frac{\mathrm{d}s}{h^3(s)},\quad n=0 \text{ or } 1. \tag{2.70}$$

When we substitute the estimates (2.60), (2.68) and (2.70) into (2.58) and use

$$a_n\geq 4+nC(B_2+2B_1)^2\log 6 \quad \text{(see (2.49), (2.50), (2.42))},$$

$$\prod_{n\geq 0}\exp -40a_n^{-3}\geq \exp -40\sum_{n=0}^{\infty}\{4+nC(B_2+2B_1)^2\log 6\}^{-3}>0$$

we obtain for $N>1$

$$\prod_0^{(N-1)+}\int_{y_{n+1}\in\Lambda_{n+1}}\mathbf{P}^{y_n}\{m_n<W^0(s)<M_n,\ 0\leq s<\lambda_n,\ W^0(\lambda_n)\in\mathrm{d}y_{n+1}\}\geq$$

$$\geq \sin\frac{\pi(y_0-m_0)}{\Delta_0}\exp -\tfrac{1}{2}\sigma^2\pi^2\int_{a_0}^{T}\frac{\mathrm{d}s}{h^2(s)}$$

$$\cdot\exp -\left[(K_{19}\wedge\varepsilon(0))\log(T+2)+K_{17}\left\{1+\int_{a_0}^{T}\frac{\varepsilon(s)}{h^2(s)}\mathrm{d}s+\int_{a_0}^{T}\frac{B_1}{h^3(s)}\mathrm{d}s\right\}\right]$$

$$\cdot\prod_0^{N-2}\int_{y_{n+1}\in\Lambda_{n+1}}\frac{2}{\Delta_n}\sin\frac{\pi(y_{n+1}-m_n)}{\Delta_n}\sin\frac{\pi(y_{n+1}-m_{n+1})}{\Delta_{n+1}}\mathrm{d}y_{n+1}$$

$$\cdot\frac{2}{\Delta_{N-1}}\int_{y_N\in\Lambda_N}\sin\frac{\pi(y_N-m_{N-1})}{\Delta_{N-1}}\mathrm{d}y_N. \tag{2.71}$$

We claim that this result remains valid for $N=0$ or $N=1$ (but $a_0<T$), provided we interpret the product of the integrals over y_{n+1}, $0\leq n\leq N-1$, as one in this case. Indeed if $N=0$ or 1, the left hand side of (2.71) consists of one factor which integrates out to

$$\mathbf{P}^{y_0}\{m_0<W^0(s)<M_0,\ 0\leq s\leq T-a_0\}\geq$$

$$\geq\left[1-5\left\{1-\exp -\frac{(T-a_0)\sigma^2\pi^2}{2\Delta_0^2}\right\}^{-2}\exp -\frac{3(T-a_0)\sigma^2\pi^2}{2\Delta_0^2}\right]$$

$$\cdot\sin\frac{\pi(y_0-m_0)}{\Delta_0}\exp -\frac{(T-a_0)\sigma^2\pi^2}{2\Delta_0^2}\frac{2}{\Delta_0}\int_{m_0}^{M_0}\sin\frac{\pi(y_1-m_0)}{\Delta_0}\mathrm{d}y_1$$

(cf. Lemma 2.4)

$$\geq\left[1-5\left\{1-\exp -\frac{(T-a_0)\sigma^2\pi^2}{2\Delta_0^0}\right\}^{-2}\exp -\frac{3(T-a_0)\sigma^2\pi^2}{2\Delta_0^2}\right]$$

$$\cdot 2\sin\frac{\pi(y_0-m_0)}{\Delta_0}\exp -\frac{(T-a_0)\sigma^2\pi^2}{2\Delta_0^2}\int_0^1\sin\pi z\,\mathrm{d}z. \tag{2.72}$$

For $N=1$ and $n=0$ (2.60), (2.63) and (2.65) remain valid and we arrive at the estimate (2.70) for

$$\left|\frac{\lambda_0}{\Delta_0^2}-\int_{I_0}\frac{\mathrm{d}s}{h^2(s)}\right| \tag{2.73}$$

as before, so that (2.71) needs no change. For $n=N=0$ we still arrive at the estimate (2.70) for (2.73) (we now can even use (2.64) instead of (2.65)), but (2.60) breaks down. As long as

$$\lambda_0=T-a_0\geqslant\frac{8\Delta_0^2}{\sigma^2\pi^2} \tag{2.74}$$

we still have

$$1-5\left\{1-\exp-\frac{\lambda_0\sigma^2\pi^2}{2\Delta_0^2}\right\}^{-2}\exp-\frac{3\lambda_0\sigma^2\pi^2}{2\Delta_0^2}\geqslant K_{20}>0$$

and we can proceed to (2.71) as before. If (2.74) fails we simply use

$$\mathbf{P}^y\{m_0<W^0(s)<M_0,\,0\leqslant s\leqslant T-a_0\}\geqslant$$

$$\geqslant\mathbf{P}^y\left\{m_0<W^0(s)<M_0,\,0\leqslant s\leqslant\frac{8\Delta_0^2}{\sigma^2\pi^2}\right\},$$

and as in (2.72) with $T-a_0$ replaced by $8\Delta_0^2/\sigma^2\pi^2$ we now obtain that this is bounded below by

$$K_{20}\sin\frac{\pi(y_0-m_0)}{\Delta_0}\mathrm{e}^{-4}2\int_0^1\sin\pi z\,\mathrm{d}z\geqslant$$

$$\geqslant\sin\frac{\pi(y_0-m_0)}{\Delta_0}\exp\left\{-\frac{\sigma^2\pi^2}{2}\int_{a_0}^T\frac{\mathrm{d}s}{h^2(s)}-K_{17}\right\}.$$

Thus, again (2.71) holds.

It is clear from (2.71) that the next step should be to estimate

$$\frac{2}{\Delta_n}\int_{y_{n+1}\in\Lambda_{n+1}}\sin\frac{\pi(y_{n+1}-m_n)}{\Delta_n}\sin\frac{\pi(y_{n+1}-m_{n+1})}{\Delta_{n+1}}\mathrm{d}y_{n+1}\quad\text{for } 0\leqslant n\leqslant N-2. \tag{2.75}$$

We begin with a bound on $|m_n-m_{n+1}|$. From the definition of m and (2.56) we have for $0\leqslant n\leqslant N-2$,

$$|m_n-m_{n+1}|\leqslant\sup_{\substack{s'\in I_n\\ s''\in I_{n+1}}}|F(s')-F(s'')|$$

$$\leqslant\sup_{s'\in I_n}|F(s')-F(a_{n+1})|+\sup_{s''\in I_{n+1}}|F(s'')-F(a_{n+1})|$$

$$\leqslant 2\sup_{s'\in \bar I_n}|F(s')-F(a_n)|+\sup_{s''\in I_{n+1}}|F(s'')-F(a_{n+1})|\ (\text{since } a_{n+1}\in\bar I_n)$$

$$\leqslant\sqrt{2}C\varepsilon(a_n)\{2H(a_n)+H(a_{n+1})\}$$

$$\leqslant 6C\varepsilon(a_n)H(a_n)\quad(\text{by (2.54), (2.55)}).$$

The same estimate holds for $|M_n - M_{n+1}|$ and[7]

$$\begin{aligned} &|\Lambda_{n+1}\Delta(m_n, M_n)| \leqslant 12C\varepsilon(a_n)H(a_n), \\ &|\Delta_n - \Delta_{n+1}| \leqslant 12C\varepsilon(a_n)H(a_n). \end{aligned} \tag{2.76}$$

Also for $y_{n+1} \in \Lambda_{n+1}$

$$\begin{aligned} &\left|\sin\frac{\pi(y_{n+1}-m_{n+1})}{\Delta_{n+1}} - \sin\frac{\pi(y_{n+1}-m_n)}{\Delta_n}\right| \leqslant \\ &\leqslant \pi\left|\frac{y_{n+1}-m_{n+1}}{\Delta_{n+1}} - \frac{y_{n+1}-m_n}{\Delta_n}\right| \leqslant \pi\Delta_n^{-1}\{|\Delta_n - \Delta_{n+1}| + |m_n - m_{n+1}|\} \\ &\leqslant 18\pi C\varepsilon(a_n)\frac{H(a_n)}{\Delta_n}. \end{aligned} \tag{2.77}$$

Taking into account that

$$\frac{2}{\Delta_n}\int_{y_{n+1}\in(m_n,M_n)} \sin^2\frac{\pi(y_{n+1}-m_n)}{\Delta_n}\,\mathrm{d}y_{n+1} = 1,$$

we see from (2.76) and (2.77) that (2.75) is at least

$$\begin{aligned} 1 - 100\pi C\varepsilon(a_n)\frac{H(a_n)}{\Delta_n} &\geqslant 1 - 100\pi C B_2^{-1}(2B_2 + 4B_1)\varepsilon(a_n) \quad \text{(see (2.61))} \\ &\geqslant \exp -200\pi C B_2^{-1}(2B_2 + 4B_1)\varepsilon(a_n) \quad \text{(see (2.53))}. \end{aligned} \tag{2.78}$$

Since in the estimate (2.68) we have already an error term of the order $\varepsilon(a_n)\log(a_{n-1}+2)$, the error due to replacing (2.75) by one can be taken care of simply by increasing K_{17}. Finally

$$\frac{2}{\Delta_{N-1}}\int_{y_N\in\Lambda_N} \sin\frac{\pi(y_N - m_{N-1})}{\Delta_{N-1}}\,\mathrm{d}y_N = 2\int_0^1 \sin\pi z\,\mathrm{d}z = \frac{4}{\pi}. \tag{2.79}$$

(2.57), (2.71), (2.78) and (2.79) together show

$$\begin{aligned} &\mathbf{P}^x\{f(s) \leqslant W^0(s) \leqslant g(s);\ 0 \leqslant s \leqslant T\} \geqslant \\ &\geqslant \int_{\Lambda_0} \mathbf{P}^x\{f(s) \leqslant W^0(s) \leqslant g(s);\ 0 \leqslant s \leqslant a_0,\ W^0(a_0) \in \mathrm{d}y_0\} \\ &\quad\cdot \sin\pi\frac{y_0 - m_0}{\Delta_0}\exp\left[-\frac{\sigma^2\pi^2}{2}\int_{a_0}^T \frac{\mathrm{d}s}{h^2(s)} - (K_{16} \wedge \varepsilon(0))\log(T+2)\right. \\ &\qquad\qquad \left. - K_{17}\left\{1 + \int_{a_0}^T \frac{\varepsilon(s)}{h^2(s)}\,\mathrm{d}s + \int_{a_0}^T \frac{B_1}{h^3(s)}\,\mathrm{d}s\right\}\right]. \end{aligned}$$

[7] In (2.76) we use $|A|$ to denote the Lebesgue measure of the set A.

Step 3. The conclusion of step 2 shows that the lower bound in (2.45) will be completely proven if we can show

$$\int_{\Delta_0} \mathbf{P}^x\{f(s)< W^0(s)<g(s), 0\leqslant s\leqslant (a_0\wedge T),$$
$$W^0(a_0\wedge T)\in \mathrm{d}y_0\}\sin \pi\frac{y_0-m_0}{\Delta_0}\,\mathrm{d}y_0 \geqslant \tag{2.80}$$
$$\geqslant K_{21}\exp -K_{22}\int_0^{a_0\wedge T}\frac{\varepsilon(s)}{h^2(s)}\,\mathrm{d}s$$

for some $0<K_{21}$, $K_{22}<\infty$, and $x\in[F(0)+B_1+\frac{1}{4}(H(0)-2B_1),\ G(0)-B_1-\frac{1}{4}(H(0)-2B_1)]$. This obviously also suffices in case $a_0\geqslant T$ if we interpret Δ_0 as $(f(T), g(T))$ and drop the factor $\sin \pi(y_0-m_0)\Delta_0^{-1}$ in this case. Note that if $a_0<T$ we have (compare the proof of (2.76))

$$\begin{aligned}\Delta_0 &\supset \left(m_0+\frac{5}{\sqrt{2}}C\varepsilon(a_0)H(a_0),\ M_0-\frac{5}{\sqrt{2}}C\varepsilon(a_0)H(a_0)\right)\\ &\supset (m_0+\tfrac{1}{8}\Delta_0,\ M_0-\tfrac{1}{8}\Delta_0)\\ &\supset (F(a_0)+B_1+\tfrac{1}{4}(H(a_0)-2B_1),\ G(a_0)-B_1-\tfrac{1}{4}(H(a_0)-2B_1)).\end{aligned}$$

Thus, by (2.41) the left hand side of (2.80) is at least

$$\sin\frac{\pi}{8}\mathbf{P}^x\{F(s)+B_1< W^0(s)< G(s)-B_1, 0\leqslant s\leqslant a,$$
$$F(a)+B_1+\tfrac{1}{4}(H(a)-2B_1)< W^0(a)< G(a)-B_1-\tfrac{1}{4}(H(a)-2B_1)\}. \tag{2.81}$$

where we have written a for $a_0\wedge T$. Moreover, by (2.49) we have on $s\leqslant a-4$,

$$\varepsilon(s)\geqslant(200\pi C)^{-1}B_2(2B_2+4B_1)^{-1}$$

and for $a-4\leqslant s\leqslant a$, $h^{-2}(s)\leqslant B_2^{-2}$ (see (2.42)). It therefore suffices to prove that (2.81) is at least

$$K_{23}\exp -K_{24}\int_0^a\frac{\mathrm{d}s}{h^2(s)}.$$

This bound will now be derived from (2.42)–(2.44) and Lemma 2.4. Since $|F'(s)|+|G'(s)|\leqslant B_3(B_2+2B_1)^{-1}(\log 2)^{-1}$ we have for any $0\leqslant u\leqslant u_0=\frac{1}{16}B_2B_3^{-1}(B_2+2B_1)\log 2$

$$|F(t+u)-F(t)|\leqslant\tfrac{1}{16}B_2,\qquad |G(t+u)-G(t)|\leqslant\tfrac{1}{16}B_2.$$

In particular, for any such u and

$$y\in(F(t)+B_1+\tfrac{1}{4}(H(t)-2B_1),\qquad G(t)-B_1-\tfrac{1}{4}(H(t)-2B_1)) \tag{2.82}$$

we have (see (2.42))

$$F(t+u)+B_1 \leq F(t)+B_1+\tfrac{1}{16}B_2 \leq y-\tfrac{3}{16}(H(t)-2B_1)$$

and

$$G(t+u)-B_1 \geq y+\tfrac{3}{16}(H(t)-2B_1).$$

Consequently for any y satisfying (2.82) and $0 \leq u_1 \leq u_0$

$$\begin{aligned}
&\mathbf{P}^y\{F(t+u)+B_1 < W^0(u) < G(t+u)-B_1,\ 0 \leq u \leq u_1,\\
&\quad F(t+u_1)+B_1+\tfrac{1}{4}(H(t+u_1)-2B_1) \leq W^0(u_1)\\
&\qquad\qquad \leq G(t+u_1)-B_1-\tfrac{1}{4}(H(t+u_1)-2B_1)\}\\
&\geq \mathbf{P}^0\{|W^0(u)| < \tfrac{3}{16}(H(t)-2B_1),\ 0 \leq u \leq u_1,\\
&\qquad F(t+u_1)+B_1+\tfrac{1}{4}(H(t+u_1)-2B_1)-y \leq W^0(u_1)\\
&\quad \leq G(t+u_1)-B_1-\tfrac{1}{4}(H(t+u_1)-2B_1)-y\}.
\end{aligned} \tag{2.83}$$

In the right hand side of (2.83) $W^0(u_1)$ is restricted to an interval of length

$$\tfrac{3}{4}(H(t+u_1)-2B_1) \geq \tfrac{3}{4}(H(t)-2B_1)-\tfrac{3}{4}\cdot\tfrac{1}{8}B_2 > \tfrac{1}{2}(H(t)-2B_1)$$

(cf. (2.42)), and with left endpoint

$$\begin{aligned}
&F(t+u_1)+B_1+\tfrac{1}{4}(H(t+u_1)-2B_1)-y \leq\\
&\quad \leq (F(t+u_1)-F(t))+\tfrac{1}{4}(H(t+u_1)-H(t)) \leq \tfrac{3}{32}B_2.
\end{aligned}$$

One shows similarly that the left endpoint of this interval is at least $-\tfrac{3}{32}B_2-\tfrac{1}{2}(H(t)-2B_1)$ so that the interval contains $\tfrac{3}{32}B_2$ or $-\tfrac{3}{32}B_2$. Therefore, the right hand side of (2.83) is at least as large as the minimum of

$$\begin{aligned}
&\mathbf{P}^0\{|W^0(u)| < \tfrac{3}{16}(H(t)-2B_1),\ 0 \leq u \leq u_1,\\
&\quad \tfrac{3}{32}B_2-\tfrac{1}{4}(H(t)-2B_1) \leq W^0(u_1) \leq \tfrac{3}{32}B_2\}
\end{aligned} \tag{2.84}$$

and

$$\begin{aligned}
&\mathbf{P}^0\{|W^0(u)| < \tfrac{3}{16}(H(t)-2B_1),\ 0 \leq u \leq u_1,\\
&\quad \tfrac{3}{32}B_2 \leq W^0(u_1) \leq \tfrac{3}{32}B_2+\tfrac{1}{4}(H(t)-2B_1)\}.
\end{aligned} \tag{2.85}$$

By virtue of Lemma 2.4, (2.83) is therefore at least

$$K_{24}\exp -K_{25}u_1(H(t)-2B_1)^{-2} \tag{2.86}$$

provided $(H(t)-2B_1)^2 \leq \tfrac{1}{4}\sigma^2\pi^2u_1$. However, if $(H(t)-2B_1)^2 > \tfrac{1}{4}\sigma^2\pi^2u_1$, then (2.84) and (2.85) are only reduced by replacing $(H(t)-2B_1)$ by $(\sigma^2\pi^2u_1/4)^{1/2}$ in which case they are still bounded below by $K_{24}\exp -4K_{25}(\sigma\pi)^{-2}$. Thus, in all cases we find that

$$\begin{aligned}
&\mathbf{P}^y\{F(t+u)+B_1< W^0(u)< G(t+u)-B_1,\ 0\leq u\leq u_1,\\
&\quad F(t+u_1)+B_1+\tfrac{1}{4}(H(t+u_1)-2B_1)\leq W^0(u_1)\\
&\qquad\qquad\qquad\qquad\leq G(t+u_1)-B_1-\tfrac{1}{4}(H(t+u_1)-2B_1)\}\\
&\geq K_{26}\exp -K_{25}u_1(H(t)-2B_1)^{-2}\\
&\geq K_{26}\exp -K_{27}\int_t^{t+u_1}\frac{\mathrm{d}s}{h^2(s)}.
\end{aligned}\tag{2.87}$$

(For the last inequality we used that for $u\leq u_0$

$$\begin{aligned}
h(t+u)&\leq H(t+u)+2B_1\leq H(t)+2B_1+B_2\\
&\leq 2H(t)\leq 2\frac{B_2+2B_1}{B_2}(H(t)-2B_1)\quad (\text{cf. (2.42)}).
\end{aligned}$$

When (2.87) is applied successively with $t=ku_0$, $u_1=u_0$ for $0\leq k<[au_0^{-1}]$ and for $t=[au_0^{-1}]u_0$, $u_1=a-[au_0^{-1}]u_0$ it is easily seen that (2.81) is bounded below by

$$\sin\frac{\pi}{8}(K_{26})^{[au_0^{-1}]+1}\exp -K_{27}\int_0^a\frac{\mathrm{d}s}{h^2(s)}.$$

Since $a=a_0\wedge T\leq B_4+4$ (see (2.44) and (2.49)) this proves (2.80).

3. Estimates for branching diffusions

Throughout this section $Z_t(\cdot)$ is the process described in the introduction. To each particle J alive at time t we associate a path P_J which is a function from $[0,t]$ to the reals,

$P_J(u)=$ position at time u of the (unique) ancestor of J alive at time u, $0\leq u\leq t$.

Let f and g be two functions from $[0,T]\to\mathbf{R}$ such that

$$0\leq f(t)\leq g(t),\quad 0\leq t\leq T,\tag{3.1}$$

and write for $t\leq T$

$$\mathscr{C}_t=\mathscr{C}_t(f,g)=\{\omega\in\Omega: f(u)<\omega(u)<g(u),\ 0\leq u\leq t\}\tag{3.2}$$

(recall that Ω denotes the set of continuous functions on $[0,\infty)$). For our second moment formulae we also use

$$\mathscr{C}^*_{s,t}=\{\omega\in\Omega: f(s+u)<\omega(u)<g(s+u),\ 0\leq u\leq t-s\},\quad 0\leq s<t\leq T.\tag{3.3}$$

Correspondingly, we define for any Borel set A,

$Z_t(A,\mathscr{C}_t)=$ number of particles J alive at time t whose position at time t is in the set A and whose associated path P_J belongs to $\mathscr{C}_t$.

$Z_t(\mathscr{C}_t)$ will sometimes be used for $Z_t(\mathbf{R},\mathscr{C}_t)$.

We also remind the reader of the following notations: $m = E\nu$ and $b = E\nu(\nu-1)$, where ν has the distribution function G, i.e., the distribution of the number of children a particle produces when it dies. $\mathbf{P}^x\{A\}$ denotes the probability of A, given that the Z-process starts with one particle at x, for sets A defined in terms of the Z-process. When A is a Brownian motion event we also use $\mathbf{P}^x\{A\}$ for the probability that a Brownian motion starting at x lies in A. It will usually be clear from the formula which interpretation applies. $\mathbf{E}^x$ denotes the expectation operator w.r.t. $\mathbf{P}^x$ in either interpretation.

Lemma 3.1. *Let $W(\cdot)$ be the Brownian motion described in Section* 2, $x > 0$, *A a Borel set and $\mathscr{C}_t$ and $\mathscr{C}^*_{s,t}$ as in* (3.1)–(3.3). *Then*

$$\mathbf{E}^x Z_t(A, \mathscr{C}_t) = e^{c(m-1)t}\mathbf{P}^x\{W(\cdot)\in\mathscr{C}_t,\ W(t)\in A\} \tag{3.4}$$

and

$$\mathbf{E}^x Z_t^2(A, \mathscr{C}_t) = \mathbf{E}^x Z_t(A, \mathscr{C}_t) + b\int_0^t e^{c(m-1)(2t-s)}\,ds \int_{-\infty}^{+\infty} \mathbf{P}^x\{W(\cdot)\in\mathscr{C}_s,\ W(s)\in dy\}$$
$$[\mathbf{P}^y\{W(\cdot)\in\mathscr{C}^*_{s,t},\ W(t-s)\in A\}]^2. \tag{3.5}$$

Proof. (3.4) and (3.5) are special cases of known formula, to wit (2.7) and (2.12) in [12]. To apply the results of [12], we merely have to make a suitable choice for the basic Markov process $\{x_t\}$ of [12]. For this we consider the continuous space time Brownian motion $\{V(t), W(t)\}_{t\geq 0}$ with state space $\mathbf{R}\times\mathbf{R}$ and transition probability

$$\begin{aligned}\mathbf{P}^{v,w}\{V(s)\in A,\ W(s)\in B\} &= \mathbf{P}\{V(t+s)\in A,\ W(t+s)\in B \mid V(t)=v,\ W(t)=w\}\\ &= I_A(v+s)\mathbf{P}^w\{W(s)\in B\} = I_A(v+s)\frac{1}{\sigma\sqrt{2\pi s}}\int_B e^{-(y-w+\mu s)^2/2\sigma^2 s}\,dy.\end{aligned}$$

We set $g(t) = +\infty$, $f(t) = -\infty$ for $t > T$ and

$$\begin{aligned}\tau &= \inf\{t\geq 0\colon W(t)\geq g(V(t)) \text{ or } W(t)\leq f(V(t))\}\\ &= \inf\{t\geq 0\colon W(t)\geq g(V(0)+t) \text{ or } W(t)\leq f(V(0)+t)\}.\end{aligned}$$

Finally we adjoin a point ∂ to the state space $\mathbf{R}\times\mathbf{R}$ of the space time Brownian motion and take for $\{x_t\}$ the process killed at time τ, i.e.,

$$x(t) = \begin{cases} V(t), W(t) & \text{if } t < \tau,\\ \partial & \text{if } t\geq\tau.\end{cases}$$

Correspondingly, we view the branching particles as moving in space-time and replace the path $P_J(u)$ by

$$\bar{P}_J(u) = \begin{cases} (u, P_J(u)) & \text{if } u < \tau_J,\\ \partial & \text{if } u \geq \tau_J,\end{cases}$$

where τ_J is the value of τ for the point $u \to (u, P_J(u))$. For $A \subset \mathbf{R}$ and $\bar{P}_J(0) = (0, x_0)$, the condition $P_J(t) \in A$, $P_J \in \mathscr{C}_t$ is then equivalent to $\bar{P}_J(t) \in \{t\} \times A$, whereas

$$\mathbf{P}^{(0,x_0)}\{x_t \in \{t\} \times A,\ x_s = (s, x)\} =$$
$$= \mathbf{P}^{x_0}\{W(t) \in A,\ f(u) < W(u) < g(u),\ s \le u \le t \mid W(s) = x\}$$
$$= \mathbf{P}^{x}\{W(\cdot) \in \mathscr{C}^*_{s,t},\ W(t-s) \in A\}.$$

Thus, for $a = (a_1, a_2)$, $b = (b_1, b_2) \in \mathbf{R} \times \mathbf{R}$, the transition function $p(s, a, b)\,\mathrm{d}b$ of (2.1) in [12] should be taken zero if $b_1 \neq a_1 + s$ and

$$\mathbf{P}^{a_2}\{W(\cdot) \in \mathscr{C}^*_{a_1, a_1+s},\ W(s) \in \mathrm{d}b_2\} \quad \text{if } b_1 = a_1 + s.$$

With this choice (3.4) and (3.5) follow from (2.7) and (2.12) of [12].

Corollary 3.2. *If*

$$\mathrm{e}^{c(m-1)(t-s)}\mathbf{P}^{y}\{W(\cdot) \in \mathscr{C}^*_{s,t},\ W(t-s) \in A\} \le \Gamma(s)$$
$$\text{for } 0 \le s \le t,\ f(s) < y < g(s), \tag{3.6}$$

and some function Γ, then

$$\mathbf{E}^{x} Z_t^2(A, \mathscr{C}_t) \le \mathbf{E}^{x} Z_t(A, \mathscr{C}_t)\left\{1 + b\int_0^t \Gamma(s)\,\mathrm{d}s\right\} \tag{3.7}$$

and

$$\mathbf{P}^{x}\{Z_t(A, \mathscr{C}_t) > 0\} \ge \left\{1 + b\int_0^t \Gamma(s)\,\mathrm{d}s\right\}^{-1} \mathbf{E}^{x} Z_t(A, \mathscr{C}_t). \tag{3.8}$$

Proof. (3.5) and (3.6) show that

$$\mathbf{E}^{x} Z_t^2(A, \mathscr{C}_t) \le \mathbf{E}^{x} Z_t(A, \mathscr{C}_t)$$
$$+ b\,\mathrm{e}^{c(m-1)t}\int_0^t \Gamma(s)\,\mathrm{d}s \int_{-\infty}^{+\infty} \mathbf{P}^{x}\{W(\cdot) \in \mathscr{C}_s,\ W(s) \in \mathrm{d}y\}$$
$$\cdot\, \mathbf{P}^{y}\{W(\cdot) \in \mathscr{C}^*_{s,t},\ W(t-s) \in A\}.$$

(3.7) now follows from (3.4) and the relation

$$\int_{-\infty}^{+\infty} \mathbf{P}^{x}\{W(\cdot) \in \mathscr{C}_s,\ W(s) \in \mathrm{d}y\}\mathbf{P}^{y}\{W(\cdot) \in \mathscr{C}^*_{s,t},\ W(t-s) \in A\} =$$
$$= \mathbf{P}^{x}\{W(\cdot) \in \mathscr{C}_t,\ W(t) \in A\},$$

which is an immediate consequence of the Markov property for $W(\cdot)$. (3.8) is immediate from (3.7) and Marshall's inequality [3, Problem 5.3.1]; alternatively, use [9, inequality II, p. 6]).

We shall use the following terminology: for $g(u)>0$, $0\le u\le t$, we say that a path P_J crosses g during $[0,t]$ if $g(0)>P_J(0)$, but $g(u)\le P_J(u)$ for some $[0,t]$. P_J crosses g for the first time during $(t,t']$ if P_J crosses g during $[0,t']$ but not during $[0,t]$ $(t<t')$. Lastly we remind the reader that $W^0(t)=W(t)+\mu t$ is a Brownian motion with mean 0 and variance coefficient σ^2.

Proposition 3.3. *Assume*

$$\mu=\{2\sigma^2c(m-1)\}^{1/2}, \tag{3.9}$$

and let $g:[0,T]\to(0,\infty)$ *be continuously differentiable and such that for some* $A<\infty$

$$g(0)\ge 2,\ g^2(0)\le 4AT \text{ and } |g(s_2)-g(s_1)|\le Ag(s_1)^{-1}(s_2-s_1)$$
$$\text{whenever } 0\le s_2-s_1\le(4A)^{-1}g^2(s_1),\ 0\le s_1\le s_2\le T. \tag{3.10}$$

Further, let $\alpha:[0,T]\to[0,\infty]$ *be such that*

$$\mathbf{P}^t\{0<W^0(s)<g(s),\ 0\le s\le t\}\le\alpha(t)\exp-\frac{\mu}{\sigma^2}\{g(0)-g(t)\},$$
$$0\le t\le T. \tag{3.11}$$

Then there exists a constant $K_1<\infty$ *which depends on* A, μ *and* σ *only such that for* $0\le x\le g(0)$, $(4A)^{-1}g^2(0)\le t\le T$

$$\mathbf{P}^x\{\text{some particle } J \text{ is alive at time } t \text{ whose path crossed } g \text{ during } [0,t]\}\le$$
$$\le e^{-(\mu/\sigma^2)(g(0)-x)}\left\{1+K_1x\int_0^{(4A)^{-1}g^2(0)}s^{-1/2}|g'(s)|\,e^{(\mu/\sigma^2)(g(0)-g(s))}\,ds\right.$$
$$\left.+K_1x\int_{(4A)^{-1}g^2(0)}^{t}|g'(s)|\alpha(s)\,ds\right\}. \tag{3.12}$$

Also, if 0 *denotes the function which is identically zero, then for* $0<x<g(0)$, $(4A)^{-1}g^2(0)\le t\le T$ *and any interval* $I\subset(0,g(t))$ *with left endpoint* y_0,

$$\mathbf{E}^xZ_t(I,\mathscr{C}_t(0,g))\le K_1x\alpha(t)\exp-\frac{\mu}{\sigma^2}\{y_0-x+g(0)-g(t)\}. \tag{3.13}$$

Lastly, for $0<x<g(0)$, $(4A)^{-1}g^2(0)\le t\le T$,

$$\mathbf{P}^x\{Z_t>0\}\le e^{-(\mu/\sigma^2)(g(0)-x)}$$
$$\cdot\left\{1+K_1x\left[\alpha(t)\,e^{(\mu/\sigma^2)g(t)}\right.\right.$$
$$+\int_0^{(4A)^{-1}g^2(0)}s^{-1/2}|g'(s)|\,e^{(\mu/\sigma^2)(g(0)-g(s))}\,ds$$
$$\left.\left.+\int_{(4A)^{-1}g^2(0)}^{t}|g'(s)|\alpha(s)\,ds\right]\right\}. \tag{3.14}$$

Proof. Abbreviate $\mathscr{C}_s(0, g)$ to $\mathscr{C}_s$. For $s+\Delta \le t$

$$\begin{aligned}
&\mathbf{P}^x\{\text{there is a particle alive at time } t \text{ whose path first crossed } g \text{ in } (s, s+\Delta]\} \le \\
&\le \mathbf{E}^x\{\text{number of particles } J \text{ alive at time } s+\Delta \text{ whose path } P_J \text{ satisfies } 0<P_J(u)<g(u),\ 0\le u\le s, \text{ but } P_J(v)\ge g(v) \text{ for some } s<v\le s+\Delta\} \\
&\le \int_{0<y<g(s)} \mathbf{E}^x Z_s(\mathrm{d}y, \mathscr{C}_s)\mathbf{E}^y\{\text{number of particles } J \text{ alive at time } \Delta \text{ with } P_J(v)>g(s+v) \text{ for some } 0<v\le\Delta\}.
\end{aligned} \tag{3.15}$$

By (3.4)

$$\begin{aligned}
&\mathbf{E}^y\{\text{number of particles } J \text{ alive at time } \Delta \text{ with } P_J(v)\ge g(s+v) \text{ for some } 0<v\le\Delta\} = \\
&= \mathbf{E}^y Z_\Delta - \mathbf{E}^y\{\text{number of particles } J \text{ alive at time } \Delta \text{ with } P_J(v)< g(s+v),\ 0<v\le\Delta\} \\
&= \mathrm{e}^{c(m-1)\Delta}(1-\mathbf{P}^y\{W(v)< g(s+v) \text{ for } 0<v\le\Delta\}).
\end{aligned} \tag{3.16}$$

For any $\varepsilon>0$ we may choose Δ so small that $g(s+v)\ge g(s)-\varepsilon$ for $0\le v\le\Delta$. Then for any $y\le g(s)-2\varepsilon$, (3.16) is bounded by

$$\begin{aligned}
&\mathrm{e}^{c(m-1)\Delta}\mathbf{P}^y\Big\{\sup_{0\le v\le\Delta} W(v)\ge g(s)-\varepsilon\Big\} \le \\
&\le \mathrm{e}^{c(m-1)\Delta}\mathbf{P}^0\Big\{\sup_{0\le v\le\Delta} W(v)\ge\varepsilon\Big\} \le 4\exp\Big\{c(m-1)\Delta-\frac{(\varepsilon+\mu\Delta)^2}{2\sigma^2\Delta}\Big\} \\
&= o(\Delta),\quad \Delta\searrow 0.
\end{aligned} \tag{3.17}$$

On the other hand, with q as in (2.5) (for $f\equiv 0$) we have by (3.4) and the Cameron–Martin formula (see (2.11))

$$\begin{aligned}
\mathbf{E}^x Z_s(\mathrm{d}y, \mathscr{C}_s) &= \mathrm{e}^{c(m-1)s}\mathbf{P}^x\{0<W(u)<g(u),\ 0\le u\le s,\ W(s)\in\mathrm{d}y\} \\
&= \mathrm{e}^{c(m-1)s}q_s(x,y)\,\mathrm{d}y \\
&= \exp\Big\{c(m-1)s-\frac{1}{2}\Big(\frac{\mu}{\sigma}\Big)^2 s-\frac{\mu}{\sigma^2}(y-x)\Big\}q_s^0(x,y)\,\mathrm{d}y \\
&= \mathrm{e}^{-(\mu/\sigma^2)(y-x)}q_s^0(x,y)\,\mathrm{d}y \quad (\text{see } (3.9)).
\end{aligned} \tag{3.18}$$

Combining (3.15)–(3.18) we find that (3.15) is bounded by

$$\int_{g(s)-2\varepsilon<y<g(s)} e^{-(\mu/\sigma^2)(y-x)} q_s^0(x,y)\,\mathrm{d}y\ \mathrm{e}^{c(m-1)\Delta}\mathbf{P}^y\{W^0(v)-\mu v\geq g(s+v)$$

$$\text{for some } 0<v\leq\Delta\}+\mathrm{o}(\Delta)\leq$$

$$\leq \mathrm{o}(\Delta)+\exp\left\{-\frac{\mu}{\sigma^2}(g(s)-2\varepsilon-x)+c(m-1)\Delta\right\}$$

$$\cdot\int_{0<y<g(s)} q_s^0(x,y)\,\mathrm{d}y\mathbf{P}^y\{W^0(v)>g(s+v) \text{ for some } 0<v\leq\Delta\}$$

$$=\mathrm{o}(\Delta)+\exp\left\{-\frac{\mu}{\sigma^2}(g(s)-2\varepsilon-x)+c(m-1)\Delta\right\}$$

$$\cdot\,\mathbf{P}^x\{0<W^0(u)<g(u),\ 0\leq u\leq s, \text{ but } W^0(v)>g(s+v)$$

$$\text{for some } 0<v\leq\Delta\}.$$

For brevity, write

$$Q_s(x)=\mathbf{P}^x\{W^0(\cdot)\in\mathscr{C}_s\}.$$

In this notation the last inequality states

$\mathbf{P}^x$\{there is a particle alive at time t whose path first crossed g in $(s,s+\Delta]\}\leq$

$$\leq \mathrm{o}(\Delta)+\exp\left\{-\frac{\mu}{\sigma^2}(g(s)-2\varepsilon-x)+c(m-1)\Delta\right\}(Q_s(x)-Q_{s+\Delta}(x)). \tag{3.19}$$

Now take $\Delta=t/n$ for n large and sum (3.19) over $s=k\Delta$, $0\leq k\leq n-1$. We obtain that the left hand side of (3.12) is bounded by

$$\mathrm{o}_n(1)+\sum_{k=0}^{n-1}\exp\left\{-\frac{\mu}{\sigma^2}\left(g\left(\frac{k}{n}t\right)-2\varepsilon-x\right)+c(m-1)\frac{t}{n}\right\}(Q_{n^{-1}kt}(x)$$

$$-Q_{n^{-1}(k+1)t}(x)).$$

If we first let $n\to\infty$ and then $\varepsilon\downarrow 0$ we find that (3.12) is bounded by

$$-\int_{0\leq s\leq t} e^{-(\mu/\sigma^2)(g(s)-x)}\,\mathrm{d}Q_s(x)=$$

$$=-Q_s(x)\,\mathrm{e}^{-(\mu/\sigma^2)(g(s)-x)}\Big|_0^t-\frac{\mu}{\sigma^2}\int_{0\leq s\leq t} g'(s)\,\mathrm{e}^{-(\mu/\sigma^2)(g(s)-x)}Q_s(x)\,\mathrm{d}s$$

$$\leq \mathrm{e}^{-(\mu/\sigma^2)(g(0)-x)}+\frac{\mu}{\sigma^2}\int_{0\leq s\leq t}|g'(s)|\,\mathrm{e}^{-(\mu/\sigma^2)(g(s)-x)}\int q_s^0(x,y)\,\mathrm{d}y. \tag{3.20}$$

Finally, we use Proposition 2.2 (with $\mu=0$) and assumption (3.11) to obtain for $s \geq (4A)^{-1}g^2(0)$

$$\int q_s^0(x,y)\,\mathrm{d}y \leq K_3^{-1}\frac{\rho(x,0)}{\rho(1,0)}\int q_s^0(1,y)\,\mathrm{d}y$$
$$\leq K_2 x \mathbf{P}^1\{0<W^0(u)<g(u),\ 0\leq u\leq s\}$$
$$\leq K_2 x\alpha(s)\exp -\frac{\mu}{\sigma^2}(g(0)-g(s)). \tag{3.21}$$

For $s<(4A)^{-1}g^2(0)$ we use the trivial estimate

$$\int q_s^0(x,y)\,\mathrm{d}y \leq \mathbf{P}^x\{W^0(u)>0,\ 0\leq u\leq s\} =$$
$$=1-\frac{2}{\sigma\sqrt{2\pi s}}\int_{-\infty}^{-x} \mathrm{e}^{-u^2/2s\sigma^2}\,\mathrm{d}u \leq K_2\frac{x}{\sqrt{s}}. \tag{3.22}$$

(3.12) follows by substitution of the estimates (3.21) and (3.22) in (3.20).

(3.13) follows from (3.4), (3.18), (3.21) and (3.11) because

$$\mathbf{E}^x Z_t(I,\mathscr{C}_t) = \mathrm{e}^{c(m-1)t}\int_{y\in I} q_t(x,y)\,\mathrm{d}y$$
$$\leq K_3^{-1}x\int_{y\in I}\mathrm{e}^{-(\mu/\sigma^2)(y-x)}q_t^0(1,y)\,\mathrm{d}y$$
$$\leq K_3^{-1}x\,\mathrm{e}^{-(\mu/\sigma^2)(y_0-x)}\mathbf{P}^1\{0<W^0(u)<g(u),\ 0\leq u\leq t\}.$$

(3.14) is now immediate, since any particle alive at time t either has a path which crossed g or a final position in $(0,g(t))$. Thus, the left hand side of (3.14) is bounded by (3.12) plus (3.13) with $I=(0,g(t))$.

Proposition 3.4. *Let* $g:[0,T]\to[2,\infty)$ *be decreasing and satisfy* (3.10). *Again assume that* (3.9) *holds. Further, let* $\beta:[0,T]\to[0,\infty]$ *be such that*

$$\mathbf{P}^1\{0<W^0(u)<g(s+u),\ 0\leq u\leq T-s\}\leq$$
$$\leq\beta(s)\exp -\frac{\mu}{\sigma^2}(g(s)-g(T)),\quad 0\leq s\leq T, \tag{3.23}$$

and let $\gamma>0$ *be such that*

$$\mathbf{P}^1\{0<W^0(s)<g(s),\ 0\leq s\leq T\}\geq\gamma\exp -\frac{\mu}{\sigma^2}(g(0)-g(T)). \tag{3.24}$$

Then there exists a constant $K_2 < \infty$ which depends on A, μ and σ only such that for $0 < x \leq \frac{1}{2}g(0)$

$$\mathbf{P}^x\{Z_T > 0\} \geq K_2 \gamma x \exp -\frac{\mu}{\sigma^2}\{g(0) - x\}$$

$$\cdot \left\{1 + b\int_0^{s_0} \beta(s) \exp \frac{\mu}{\sigma^2} g(T)\, ds + \int_{s_0}^{T} (T-s)^{-1/2} \exp \frac{\mu}{\sigma^2} g(s)\, ds\right\}^{-1}, \quad (3.25)$$

where $s_0 = T - (2A)^{-1} g^2(T)$.

Remark 3.5. We may replace (3.23) by the following two conditions at the price of a slightly more complicated expression in the right hand side of (3.25):

$$\mathbf{P}^1\{0 < W^0(u) < g(u), 0 \leq u \leq T\} \leq \alpha \exp -\frac{\mu}{\sigma^2}\{g(0) - g(T)\}$$

and for $0 \leq t \leq T$

$$\mathbf{P}^1\{0 < W^0(u) < g(u), 0 \leq u \leq t\} \geq \gamma(t) \exp -\frac{\mu}{\sigma^2}\{g(0) - g(t)\}.$$

Indeed, these two conditions imply (3.23) with $\beta(s) = \{K_8 g(s)\gamma(s)\}^{-1}\alpha$ for $(4A)^{-1}g^2(0) \leq s \leq s_0$, by virtue of (2.39). By the end of the proof of Proposition 3.4 it suffices to have (3.23) for $s \leq s_0$. Replacing (3.23) by the above two conditions gives Propositions 3.3 and 3.4 a more symmetric appearance.

Proof. Again by (3.4), the Cameron–Martin formula and (3.9)

$$\mathbf{E}^x Z_T(\mathscr{C}_T(0, g)) = e^{c(m-1)T} e^{-(\mu^2/2\sigma^2)T}$$

$$\cdot \int_{0<y<g(T)} e^{-(\mu/\sigma^2)(y-x)} \mathbf{P}^x\{0 < W^0(s) < g(s), 0 \leq s \leq T, W^0(T) \in dy\}$$

$$\geq e^{-(\mu/\sigma^2)(g(T)-x)} \mathbf{P}^x\{0 < W^0(s) < g(s), 0 \leq s \leq T\}.$$

Proposition 2.2 (with $\mu = 0$) and (3.24) now give

$$\mathbf{P}^x\{0 < W^0(s) < g(s), 0 \leq s \leq T\} = \int q_T^0(x, y)\, dy$$

$$\geq K_3 \frac{\rho(x, 0)}{\rho(1, 0)} \int q_T^0(1, y)\, dy = K_3 x \mathbf{P}^1\{0 < W^0(s) < g(s), 0 \leq s \leq T\}$$

$$\geq K_3 \gamma x \exp -\frac{\mu}{\sigma^2}\{g(0) - g(T)\}.$$

Thus

$$\mathbf{E}^x Z_T(\mathscr{C}_T(0, g)) \geqslant K_3 \gamma x \exp -\frac{\mu}{\sigma^2}\{g(0)-x\}. \tag{3.26}$$

Next we need an estimate of the form (3.6). For $0 \leqslant s \leqslant T$, $T-s \geqslant (4A)^{-1} g^2(s)$, $0 < y < g(s)$

$$\begin{aligned}
&\mathrm{e}^{c(m-1)(T-s)} \mathbf{P}^y\{W(\cdot) \in \mathscr{C}^*_{s,T}\} = \\
&= \mathrm{e}^{c(m-1)(T-s)-(\mu^2/2\sigma^2)(T-s)} \\
&\quad \cdot \int_{0<z<g(T)} \mathrm{e}^{-(\mu/\sigma^2)(z-y)} \mathbf{P}^y\{0 < W^0(u) < g(s+u), 0 \leqslant u \leqslant T-s, \\
&\qquad\qquad W^0(T-s) \in \mathrm{d}z\} \\
&\leqslant \mathrm{e}^{(\mu/\sigma^2)y} \mathbf{P}^y\{0 < W^0(u) < g(s+u), 0 \leqslant u \leqslant T-s\} \\
&\leqslant \mathrm{e}^{(\mu/\sigma^2)y} K_8^{-2} \frac{\rho(y,s)}{\rho(1,s)} \mathbf{P}^1\{0 < W^0(u) < g(s+u), 0 \leqslant u \leqslant T-s\} \qquad (3.26) \\
&\text{(by (2.37))} \leqslant K_8^{-2}(g(s)-y)\beta(s) \exp -\frac{\mu}{\sigma^2}\{g(s)-y-g(T)\} \\
&\text{(by (3.23))} \leqslant K_4 \beta(s) \exp \frac{\mu}{\sigma^2} g(T).
\end{aligned}$$

For any $0 \leqslant s \leqslant T$, $0 < y < g(s)$ we also have (recall that g is decreasing)

$$\begin{aligned}
&\mathrm{e}^{c(m-1)(T-s)} \mathbf{P}^y\{W(\cdot) \in \mathscr{C}^*_{s,T}\} \leqslant \\
&\leqslant \mathrm{e}^{(\mu/\sigma^2)y} \mathbf{P}^y\left\{\sup_{0 \leqslant u \leqslant T-s} W^0(u) < g(s)\right\} \\
&\leqslant K_5 \,\mathrm{e}^{(\mu/\sigma^2)y}(g(s)-y)(T-s)^{-1/2} \\
&\leqslant K_6 \,\mathrm{e}^{(\mu/\sigma^2)g(s)}(T-s)^{-1/2}.
\end{aligned}$$

Lastly we observe that (3.10) implies (since g is decreasing)

$$-g'(s) = |g'(s)| \leqslant A\{g(s)\}^{-1} \quad \text{or} \quad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s} g^2(s) \geqslant -A,$$

whence

$$g^2(s) \leqslant g^2(T) + 2A(T-s).$$

Thus $T-s \geqslant (4A)^{-1} g^2(s)$ is fulfilled as long as $s \leqslant s_0$, where $s_0 = T-(2A)^{-1} g^2(T)$. We therefore have (3.6) with

$$\Gamma(s) = \begin{cases} K_4 \beta(s) \exp \dfrac{\mu}{\sigma^2} g(T) & \text{if } 0 \leqslant s \leqslant s_0, \\[2ex] K_6 \exp \dfrac{\mu}{\sigma^2} g(s)(T-s)^{-1/2} & \text{if } s_0 < s \leqslant T. \end{cases}$$

An application of (3.8) together with (3.26) now yields

$$\mathbf{P}^x\{Z_T>0\}\geq K_3\gamma x\exp -\frac{\mu}{\sigma^2}\{g(0)-x\}$$
$$\cdot\{1+b\int_0^{s_0}\beta(s)\exp\frac{\mu}{\sigma^2}g(T)\,\mathrm{d}s$$
$$+\int_{s_0}^{T}\exp\frac{\mu}{\sigma^2}g(s)(T-s)^{-1/2}\,\mathrm{d}s\}^{-1}$$

as claimed.

Proof of Theorem 1.3. It only requires some simple applications of Proposition 2.5, 3.3 and 3.4 to prove Theorem 1.3. The idea of the proof of (1.10) is to find a function g on $[0, T]$ such that

$$\mathbf{P}^1\{0<W^0(s)<g(s), 0\leq s\leq t\}\exp\frac{\mu}{\sigma^2}\{g(0)-g(t)\}$$

is "close to one" for $0\leq t\leq T$. As indicated in Remark 3.5 we can then expect also

$$\mathbf{P}^1\{0<W^0(u)<g(s+u), 0\leq u\leq T-s\}\exp\frac{\mu}{\sigma^2}\{g(s)-g(T)\}$$

to be "close to one" for $0\leq s\leq T$. We will then have (3.11) and (3.23) with $\alpha(\cdot)$ and $\beta(\cdot)$ "close to one" and by Proposition 3.3 and 3.4 $\exp -(\mu/\sigma^2)\{g(0)-x\}$ will then approximately be the order of $\mathbf{P}^x\{Z_T>0\}$. We find a function g with the above properties with the help of Proposition 2.5.

As for the details, we first apply Proposition 2.5 with $f\equiv 0$, $g(s)=\Lambda(T+D-t+s)^{1/3}$, $0\leq s\leq t$, where

$$\Lambda=\left(\frac{3\sigma^4\pi^2}{2\mu}\right)^{1/3}$$

and $0\leq t<T$ and $e^3\leq D\leq T$ are fixed. This clearly satisfies (2.41)–(2.44) with $B_1=0$ (i.e., $F\equiv 0$, $G=g$), $B_2=\Lambda$ e,

$$\varepsilon(s)=\tfrac{1}{3}\Lambda^2(T+D-t+s)^{-1/3}\log(T+D-t+s)$$

(this ε is decreasing and we can take $B_3=\Lambda^2/e$; B_4 can be chosen independent of D). Therefore, by (2.48)

$$\mathbf{P}^x\{0<W^0(s)<\Lambda(T+D-s)^{1/3}, 0\leq s\leq t\}<$$
$$\leq\sin(\pi x\Lambda^{-1}(T+D)^{-1/3})\exp\Big[-\frac{\mu}{\sigma^2}\Lambda\{(T+D)^{1/3}-(T+D-t)^{1/3}\}$$
$$+K_1(\log(T+D))^2\Big] \qquad (3.28)$$

for some $K_1<\infty$, depending on μ and σ only, provided

$$T\geq t\geq(8B_3)^{-1}(\log 2)\Lambda^2(T+D)^{2/3}.$$

We may now apply Proposition 3.3 with

$$g(t)=g(t;D)=A(T+D-t)^{1/3},\quad 0\leq t\leq T.$$

(3.28) shows that (3.11) holds with $\alpha(t)=\exp K_1(\log 2T)^2$, uniformly for $e^3\leq D\leq T$, as long as

$$T\geq t\geq(8B_3)^{-1}(\log 2)A^2(2T)^{2/3}\text{ and }AT^{1/3}\geq 2.\tag{3.29}$$

However, by raising K_1, if necessary, we have (3.11) for the above $\alpha(\cdot)$ and all $0\leq t\leq T$, $e^3\leq D\leq T$, $T\geq K_2$, because when (3.29) fails

$$\frac{\mu}{\sigma^2}\{g(0;D)-g(t;D)\}=\frac{\mu}{\sigma^2}A\{(T+D)^{1/3}-(T+D-t)^{1/3}\}$$

$$\leq\frac{\mu A}{3\sigma^2}t(T+D-t)^{-2/3}\leq K_3.$$

Again $K_2, K_3<\infty$ depend on μ, σ only. Also (3.10) is trivially satisfied for $g(\cdot\,;D)$ for $T\geq K_2$ and some A depending on μ,σ only. We therefore conclude from (3.12)–(3.14), that if (1.9) holds, there exist constants K_4, $K_5<\infty$, depending on σ, m, b and c only such that for

$$T\geq K_4,\quad 0<x<A(T+D)^{1/3},\quad e^3\leq D\leq T,$$

$\mathbf{P}^x$\{some particle J is alive at time T whose path crossed $g(\cdot\,;D)$ during $[0,T]$\} $\leq$

$$\leq(1+x)\exp-\frac{\mu}{\sigma^2}A(T+D)^{1/3}\cdot\exp\left\{\frac{\mu}{\sigma^2}x+K_5(\log 2T)^2\right\},\tag{3.30}$$

$\mathbf{E}^x$\{number of particles J alive at time T whose path did not cross $g(\cdot\,;D)$ during $[0,T]$\} $\leq$

$$\leq x\exp-\frac{\mu}{\sigma^2}A(T+D)^{1/3}\cdot\exp\left\{\frac{\mu}{\sigma^2}x+K_5(\log 2T)^2+\frac{\mu}{\sigma^2}AD^{1/3}\right\}\tag{3.31}$$

(take $I=(0,g(0;D))$ in (3.13)), and

$$\mathbf{P}^x\{Z_T>0\}\leq(1+x)\exp-\frac{\mu}{\sigma^2}AT^{1/3}\exp\left\{\frac{\mu}{\sigma^2}x+K_5(\log 2T)^2\right\}\tag{3.32}$$

(take $D=e^3$ in (3.14)). Similarly if we take $f\equiv 0$, $g(s)=A(s+e^3)^{1/3}$, $0\leq s\leq T-t$ in Proposition 2.5 then we obtain from (2.48) with t replaced by $T-t$

$$\mathbf{P}^x\{0<W^0(s)<A(T-t-s+e^3)^{1/3},0\leq s\leq T-t\}=$$

$$=\sin(\pi xA^{-1}(T-t+e^3)^{-1/3})\cdot\exp\left\{-\frac{\mu}{\sigma^2}A(T-t+e^3)^{1/3}+\theta K_1(\log 2T)^2\right\}$$

for some $|\theta|\leq 1$, provided

$$T-t\geq K_6(T-t+e^3)^{2/3}\quad\text{or}\quad(T-t)\geq K_7.$$

Almost as above we can now apply Proposition 3.4 with $g(\cdot)=g(\cdot\,;e^3)$, $\beta(t)=\exp K_1(\log 2T)^2$ and $\gamma=\sin(\pi\Lambda^{-1}(T+e^3)^{-1/3})\exp -K_1(\log 2T)^2$. This yields, for $T\geq K_4$, $0<x<\frac{1}{2}T^{1/3}$

$$\mathbf{P}^x\{Z_T>0\}\geq x\exp -\frac{\mu}{\sigma^2}\Lambda T^{1/3}\exp\left\{\frac{\mu}{\sigma^2}x-K_5(\log 2T)^2\right\}. \tag{3.33}$$

(3.32) and (3.33) prove (1.10).

To prove (1.11) and (1.12) we now take $D=D_0=K_8T^{2/3}(\log 2T)^2$ in (3.30) and (3.31), where $K_8=3\sigma^2(\mu\Lambda)^{-1}(2K_5+2)$, so that

$$\begin{aligned}(T+D_0)^{1/3}&=\{T+K_8T^{2/3}(\log 2T)^2\}^{1/3}=T^{1/3}+\tfrac{1}{3}K_8(\log 2T)^2\\ &\quad+O\{T^{-1/3}(\log 2T)^4\}\geq T^{1/3}+\frac{\sigma^2}{\mu}\Lambda^{-1}(2K_5+1)(\log 2T)^2\end{aligned}$$

for large enough T. We then find from (3.30) and (3.33), that for fixed x

$$\mathbf{P}^x\{\text{some particle } J \text{ is alive at time } T \text{ whose path crossed } g(\cdot\,;D_0) \text{ during } [0,T]\}=o(\mathbf{P}^x\{Z_T>0\}),\quad T\to\infty, \tag{3.34}$$

and by dividing (3.31) by (3.33)

$$\begin{aligned}&\mathbf{E}^x\{\text{number of particles alive at time } T \text{ whose path did not cross } g(\cdot\,;D_0) \text{ during } [0,T] \,|\, Z_T>0\}\\ &\leq\exp\left[\frac{\mu}{\sigma^2}\Lambda\{T^{1/3}-(T+D_0)^{1/3}+D_0^{1/3}\}+2K_5(\log 2T)^2\right]\\ &\leq\exp\left[\frac{\mu}{\sigma^2}\Lambda K_8^{1/3}T^{2/9}(\log 2T)^{2/3}+2K_5(\log 2T)^2\right].\end{aligned} \tag{3.35}$$

Since we may ignore the particles whose paths crossed $g(\cdot\,;D_0)$ (1.12) follows from (3.35). Also (1.11) follows from this since any particle whose path does not cross $g(\cdot\,;D_0)$ must have a position at time T in $(0,g(T;D_0))=(0,\Lambda K_8^{1/3}T^{2/9}(\log T)^{2/3})$.

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Theorem 1.3 was proven because it was originally thought that this would be useful for Dr. M. Bramson's thesis [2], which obtains very precise results for the position of the particle furthest to the right if no absorption on $(-\infty,0])$ takes place. The author is obliged to Dr. Bramson for many profitable discussions on branching diffusions.

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